ON THE NATURE OF FOUR MODELS OF SYMMETRIC WALKS AVOIDING A QUADRANT

THOMAS DREYFUS AND AMÉLIE TROTIGNON

Abstract. We study the nature of the generating series of some models of walks with small steps in the three quarter plane. More precisely, we restrict ourselves to the situation where the group is infinite, the kernel have genus one, and the step set is symmetric, with no anti-diagonal directions. In that situation, a functional equation can be solved. Among the four models of walks, we obtain, using difference Galois theory, that three of them have a differentially transcendental generating series, and one has a differentially algebraic generating series.

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Introduction

In this paper, we consider walks with small steps in the three quadrants

$$\mathcal{C} = \{(i,j) \in \mathbb{Z}^2 : i \geqslant 0 \text{ or } j \geqslant 0\}.$$

More precisely, we encode the eight cardinal directions of the plane by pairs of integers (i,j) with $i,j \in \{0,\pm 1\}$. We consider models of walk in the three quarter plane $\mathcal{C}$ satisfying the following properties:

- the walks start at $(0,0)$;
- the walks take their step in $\mathcal{S} \subset \{0,\pm 1\}^2$. The latter is called the step set.

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We introduce the corresponding trivariate generating series

$$C(x, y; t) := \sum_{n=0}^{\infty} \sum_{(i,j)\in\mathcal{C}} c_{i,j}(n) x^i y^j t^n,$$

where $c_{i,j}(n)$ denotes the number of walks in $\mathcal{C}$ reaching the position (i, j) from the initial position $(0, 0)$ after n steps in $\mathcal{S}$. We stress out the fact that the paper is written in a more general weighted context, see Section 1, but to simplify the exposition, we are going to restrict ourselves to the unweighted case in the introduction.

Context. Enumeration of lattice walks is a central question in combinatorics. Most studies have been done on walks confined in convex cones with various methods and technics which reinforce the other in many ways : combinatorics [MR09, BMM10, MM14, MM16], complex analysis [FIM17, KR11, Ras12, FR12, KR12, KR15], probability theory [DW15], computer algebra [BK10, BCVH+17], Galois theory of difference equations [DHRS18, DHRS20b, DR19, DH19]. Three natural questions arise in walks studies: exact expressions for the generating function of the number of walks, computation of the asymptotic behavior of the number of excursions, and the nature of the trivariate generating function. In this article, we are interested in the nature of the generating function $C(x, y; t)$ of walks avoiding a quadrant. A function can be rational, algebraic, D-finite, D-algebraic or D-transcendental with the following hierarchical chain

$$\text{rational} \quad \subsetneq \quad \text{algebraic} \quad \subsetneq \quad \text{D-finite} \quad \subsetneq \quad \text{D-algebraic.}$$

Rational and algebraic functions are classical notions. By $C(x, y; t)$ D-finite (resp. D-algebraic) we mean that all of $x \mapsto C(x, y; t)$, $y \mapsto C(x, y; t)$, $t \mapsto C(x, y; t)$ satisfy a nontrivial linear (resp. algebraic) differential equation with coefficients in $\mathbb{Q}(x, y, t)$. We say that $C(x, y; t)$ is D-transcendental if it is not D-algebraic. We refer to Section 3 for more details.

Walks in the plane always have rational generating functions, while walks in the half plane always have algebraic generating functions. The next step is to consider walks in the quarter plane. The situation gets much more complicated, and the nature of the generating series depends on the choice of the step set $\mathcal{S}$. This question has generated a great interest and the determination of the nature of the generating series is now complete, see Figure 1. More precisely, the study has been started with the seminal paper [BMM10]. In this article, the authors prove that after elimination of trivial or one dimensional cases, and after considering symmetries, only 79 cases remain to study over the original $2^8 = 256$ possible step sets $\mathcal{S}$. They introduced a notion of group associated to the step set, and using combinatorial methods, prove that over the 23 step sets with finite group, 22 have a generating series which is D-finite, and even algebraic in 3 cases. With computer algebra, the last finite group case was solved in [BK10] (see also [FR10, MW20]), and proved to be algebraic. The study of the 56 step sets with infinite group is more tricky. An algebraic curve is associated with the step set which has genus one in 51 cases and genus zero in the 5 other cases. With complex analysis, the authors of [KR12] proved that in the genus one case with infinite group, the generating series is not D-finite. Their work is based in the study of the uniformization of an elliptic curve initiated in [FIM17]. The classification between D-algebraic and D-transcendental is more recent. In [BBMR17], it is proved that in the 51 above cited step sets, 9 have a generating series which is D-algebraic. The difference Galois theory, see [DHRS18, DHRS20b, DH19], allows to prove that the other 47 step sets lead to D-transcendental generating series. This complete the study of the nature of the generating series of walks in the quarter plane case. We refer to [BBMM18] for a starting point of the study of walks with large steps, and for instance in [DR19], for generalization of some results in the weighted context.

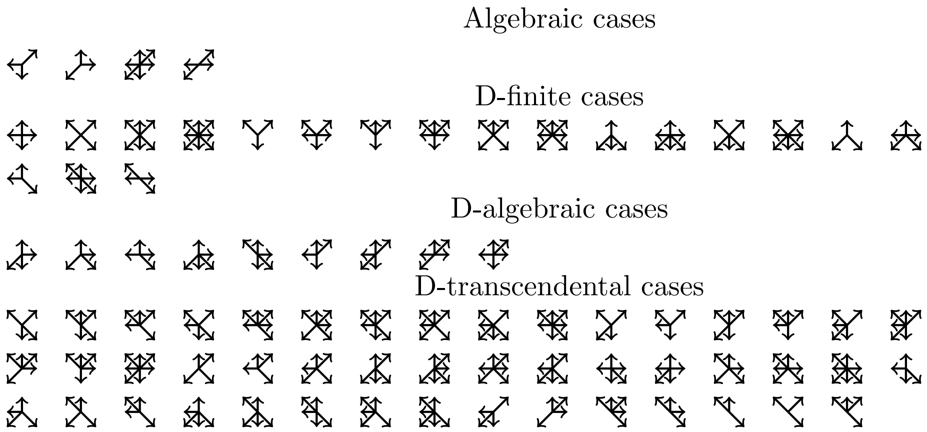


FIGURE 1. Classification of the 79 models of walks in the quarter plane. The algebraic and D-finite cases correspond to walks with a finite group.

The next step is the study of walks in the three quarter plane. This question is not vacuous, since every walks confined in a cone is equivalent to a walk, with possibly large steps, confined to the half plane, quarter plane (when the cone is convex), or three quarter plane (when the cone is concave). Similarly to the quarter plane case, the study of walks with large steps is far away to be reachable in the full generality. Let us focus on the small steps case. The 5 genus zero configurations, which correspond to the situation where the step set is included in , are not concerned with the three quarter plane restriction, and therefore have a rational generating series like every walks in the plane. In [BM16], Bousquet-Mélou proves that the simple walks (with step set) and the diagonal (with step set) walks have a D-finite generating function. Few years after, with the same methods, Bousquet-Mélou and Wallner [BMW20] prove that the king walk (with step set) also has a D-finite generating function. In [Mus19], Mustapha proves that the generating function for genus one walks with infinite group is not D-finite. Finally, in [RT19], using integral expression of the generating function, the authors prove that four symmetric models with finite group have a D-finite generating series. Figure 2 summarizes the current classification of walks in the three-quarter plane.

Method and main result. As mentioned in [RT19, Tro19], the study of walks avoiding a quadrant gives rise to convergence problems. Indeed, although we can easily write a functional equation for the generating function $C(x, y; t)$, unlike the quadrant case, see

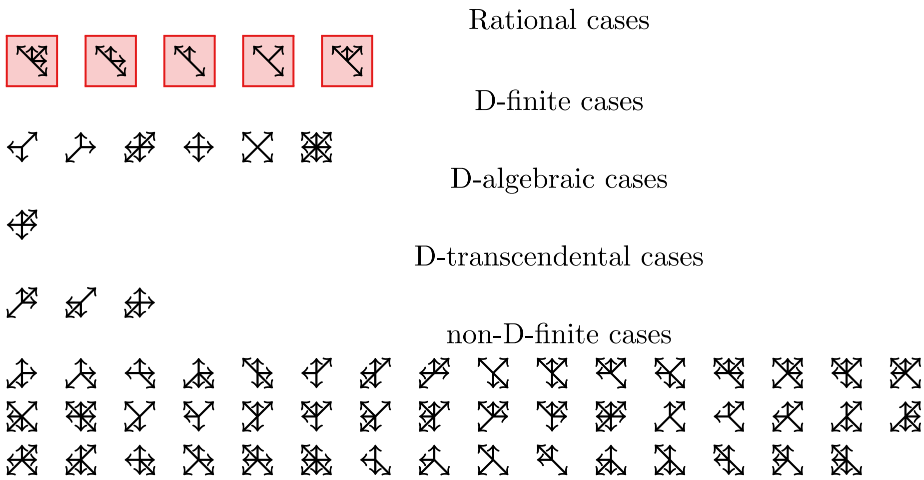


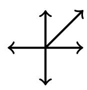
FIGURE 2. Current classification of some models of walks in the three-quarter plane. Except the 5 rational cases, the D-finite cases correspond to walks with a finite group. The models framed in red have a different nature in the quarter plane case and in the three quarter plane case.

FIGURE 3. Symmetric step sets with infinite group and no anti-diagonal directions.

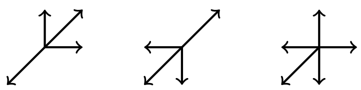
[BMM10, Sec. 4.1], this functional equation involves infinitely many negative and positive power of x and y making the series not convergent anymore. In this article, we follow the same strategy as [RT19, Tro19] and divide the three quadrants into two symmetric convex cones of opening angle $3\pi/4$ and the diagonal in between, see Figure 4. After making some assumptions on the step set of the walk (symmetry and no anti-diagonal jumps in (H1)), we derive a functional equation which can be solved. We are then able to use the tools and techniques of [DHRS18] to find the nature of the generating function for the models of Figure 3. More precisely, we prove the following, see Theorem 3.12.

Theorem. *The following holds:*

- *Assume that the step set of the walk is the following:*

  Then, $C(x, y; t)$ is D-algebraic.
- *Assume that the step set of the walk is one of the following:*

  Then, $C(x, y; t)$ is D-transcendental. More precisely, let $|\mathcal{S}|$ denotes the cardinal of $\mathcal{S}$. For any fixed $t \in]0, 1/|\mathcal{S}|[$, the series $x \mapsto C(x, y; t)$ (resp. $y \mapsto C(x, y; t)$)

does not satisfy any nontrivial algebraic differential equation in coefficients in $\mathbb{Q}(x, y, t)$.

We stress out the fact that for the four models of Figure 3, the generating series has same nature in the quarter plane case and in the three quarter plane case.

Structure of the paper. In Section 1, we start with the introduction of an important object called the kernel of the walk denoted by $K(x, y)$. It is a polynomial of degree 2 in x and in y which encodes the step set of the walk. We further derive various functional equations satisfied by the generating functions of the walks involving the kernel. Finally, after presenting some properties of the kernel curve, we recall the notion of the group of the walk. This group only depends on the steps of the walk and is generated by two bi-rational transformations. In Section 2, we first uniformize the elliptic curve defined by the zeros of the kernel. We also continue analytically the generating functions on the kernel curve and then after parametrization, on the whole complex plane $\mathbb{C}$. In Section 3, we study the nature of generating series corresponding to the four step sets of Figure 3.

1. Kernel and functional equations

1.1. **Weighted walks.** In this paper, we are going to consider a more general context of weighted walks. More precisely, let $(d_{i,j})_{(i,j)\in\{0,\pm1\}^2}$ be a family of elements of $\mathbb{Q} \cap [0, 1]$ such that $\sum_{i,j} d_{i,j} = 1$. We consider weighted models of walk in the three quarter plane $\mathcal{C}$ satisfying the following properties:

- the walks start at $(0, 0)$;
- for $(i, j) \in \{0, \pm1\}^2\backslash\{(0, 0)\}$ (resp. $(0, 0)$), the element $d_{i,j}$ is a weight on the step (i, j) and can be viewed as the probability for the walk to go in the direction (i, j) (resp. to stay at the same position).

If $d_{0,0} = 0$ and if the nonzero $d_{i,j}$ all have the same value, we say that the model is called unweighted.

The set of steps of the weighted walk is the set of cardinal directions with nonzero weight, that is,

$$\mathcal{S} = \{(i, j) \in \{0, \pm1\}^2 | d_{i,j} \neq 0\}.$$

The *weight of the walk* is defined to be the product of the weights of its component steps. From now on, we replace the definition of $c_{i,j}(n)$ by the sum of the weights of all walks reaching the position $(i, j) \in \mathcal{C}$ from the initial position $(0, 0)$ after n steps. We also replace $C(x, y; t)$ by the following trivariate generating series

$$C(x, y; t) := \sum_{n=0}^{\infty} \sum_{(i,j)\in\mathcal{C}} c_{i,j}(n) x^i y^j t^n.$$

Remark 1.1. *Consider an unweighted walk with generating series* $C(x, y; t)$. *Then, the series* $C(x, y; t|\mathcal{S}|)$, *represents the generating series in the introduction. We stress out the fact that the latter has the same nature as* $C(x, y; t)$, *so that this abuse of notations has no importance in what follows.*

From now on, we fix $0 < t < 1$. To simplify notations, we may sometimes omit the dependance in t. In particular, $C(x, y; t)$ will also be denoted by $C(x, y)$.

1.2. **Kernel of the walk.** Usually, the starting point is to reduce the enumerating problem of walks to the resolution of a functional equation. Before deriving this equation satisfied by the generating function $C(x, y)$, let us present the kernel, a crucial object, which depends on the step set.

The polynomial

$$K(x,y) = xy \left[t \sum_{(i,j)\in\mathcal{S}} d_{i,j} x^i y^j - 1 \right] \tag{1}$$

is called the kernel of the walk. It encodes the elements of $\mathcal{S}$ (the steps of the walks). We can rewrite it as:

$$K(x,y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y), \tag{2}$$

where:

$$\begin{cases} a(x) = tx \displaystyle\sum_{i\in\{0,\pm1\}} d_{i,1}x^i; & b(x) = tx \displaystyle\sum_{i\in\{0,\pm1\}} d_{i,0}x^i - x; & c(x) = tx \displaystyle\sum_{i\in\{0,\pm1\}} d_{i,-1}x^i; \\ \widetilde{a}(y) = ty \displaystyle\sum_{j\in\{0,\pm1\}} d_{1,j}y^j; & \widetilde{b}(y) = ty \displaystyle\sum_{j\in\{0,\pm1\}} d_{0,j}y^j - y; & \widetilde{c}(y) = ty \displaystyle\sum_{j\in\{0,\pm1\}} d_{-1,j}y^j. \end{cases} \tag{3}$$

We also define the discriminants in x and y of the kernel (1):

$$\widetilde{d}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y) \quad \text{and} \quad d(x) = b(x)^2 - 4a(x)c(x). \tag{4}$$

Let us fix a determination of the complex square root. Let $Y_\pm(x)$ (resp. $X_\pm(y)$) be the algebraic functions defined by the relation $K(x, Y_\pm(x)) = 0$ (resp. $K(X_\pm(y), y) = 0$). Obviously with (2) and (4) we have

$$Y_\pm(x) = \frac{-b(x) \pm \sqrt{d(x)}}{2a(x)}, \quad \text{and} \quad X_\pm(y) = \frac{-\widetilde{b}(y) \pm \sqrt{\widetilde{d}(y)}}{2\widetilde{a}(y)}. \tag{5}$$

1.3. **Functional equations.** From now on, we assume the following hypothesis:

(H1) For all $(i, j) \in \mathcal{S}$, $d_{i,j} = d_{j,i}$ and $d_{1,-1} = d_{-1,1} = 0$.

As in the case of the generating function of quadrant walks [BMM10, Sec. 4.1], by a recursive construction of a walk by adding a new step at the end of the walk at each stage, we can easily derive a first functional equation for $C(x, y)$, see [BM16, Sec. 2] and [RT19, Sec. 2.1]. We obtain

$$\begin{aligned} C(x,y) = 1 + t \sum_{(i,j)\in\mathcal{S}} d_{i,j}x^i y^j C(x,y) - t \sum_{i\in\{0,\pm1\}} d_{i,-1}x^i y^{-1} C_{-0}(x^{-1}) \\ - t \sum_{j\in\{0,\pm1\}} d_{-1,j}x^{-1}y^j C_{0-}(y^{-1}) - td_{-1,-1}x^{-1}y^{-1}C_{0,0}, \end{aligned} \tag{6}$$

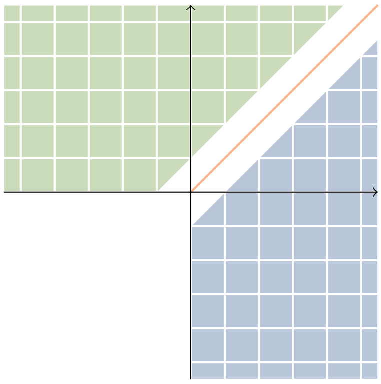


FIGURE 4. Splitting of the three-quadrant cone in two wedges of opening angle $\frac{3\pi}{4}$. Green part corresponds to U, blue part to L, and orange part to D.

where

$$C_{-0}(x^{-1}) = \sum_{n\geqslant 0, i>0} c_{-i,0}(n)x^{-i}t^n,\ C_{0-}(y^{-1}) = \sum_{n\geqslant 0, j>0} c_{0,-j}(n)y^{-j}t^n,\ C_{0,0} = \sum_{n\geqslant 0} c_{0,0}(n)t^n.$$

Multiplying both sides by $-xy$ we get

$$K(x,y)C(x,y) = c(x)C_{-0}(x^{-1}) + \widetilde{c}(y)C_{0-}(y^{-1}) + td_{-1,-1}C_{0,0} - xy. \tag{7}$$

Because of the presence of infinitely many terms with positive and negative valuations in x or y, the resolution of the functional equation (6) by algebraic substitutions or evaluations at well-chosen complex points seems complicated, as the series are no longer convergent. Therefore, we use the same strategy as in [RT19]: we split the three-quadrant into three parts and decompose the generating function $C(x,y)$ into a sum of three generating functions

$$C(x,y) = L(x,y) + D(x,y) + U(x,y), \tag{8}$$

where

$$L(x,y) = \sum_{\substack{i\geqslant 0\\ j\leqslant i-1\\ n\geqslant 0}} c_{i,j}(n)x^iy^jt^n, \quad D(x,y) = \sum_{\substack{i\geqslant 0\\ n\geqslant 0}} c_{i,i}(n)x^iy^it^n, \quad U(x,y) = \sum_{\substack{j\geqslant 0\\ i\leqslant j-1\\ n\geqslant 0}} c_{i,j}(n)x^iy^jt^n.$$

The generating function $L(x,y)$ (resp. $D(x,y)$ and $U(x,y)$) represents walks ending in the lower cone $\{i \geqslant 0, j \leqslant i-1\}$ (resp. the diagonal $\{i = j\}$ and the upper cone $\{j \geqslant 0, i \leqslant j-1\}$), see Figure 4.

Thanks to the symmetry of the step set and the fact that the starting point $(0,0)$ lies on the diagonal, we have $U(x,y) = L(y,x)$ and $C(x,y)$ can be written as the sum $L(x,y) + D(x,y) + L(y,x)$ of two unknown generating functions.

Lemma 1.2. *For any step set which satisfies (H1) and any walk that starts at* $(0,0)$*, one has*

$$
\begin{aligned}
(9)\quad K(x,y)L(x,y) &= -\frac{xy}{2} + \widetilde{c}(y)C_{0-}(y^{-1}) + \frac{t}{2}d_{-1,-1}C_{0,0} \\
&- xy\left(-\frac{1}{2} + t\left(\frac{1}{2}(d_{1,1}xy + d_{0,0} + d_{-1,-1}x^{-1}y^{-1}) + d_{0,-1}y^{-1} + d_{1,0}x\right)\right)D(x,y).
\end{aligned}
$$

Proof. This may be straightforwadly deduced from the proof of [RT19, Appx. C]. □

As in [RT19], let us perform on (9) the change of coordinates

$$
(10)\qquad \varphi(x,y) = (xy, x^{-1}).
$$

Note that it is bijective with inverse $\varphi^{-1}(x,y) = (y^{-1}, xy)$. This change of coordinates change the lower octant into the positive quadrant (see Figure 5). If we also multiply the functional equation (9) by x, we get by [RT19, (15)], the functional equation

$$
(11)\qquad \widetilde{K}_\varphi(x,y)L_\varphi(x,y) = xf(x)C_{0-}(x) + xg(x,y)D_\varphi(y) + \frac{t}{2}d_{-1,-1}xC_{0,0} - \frac{xy}{2},
$$

where

$$
\begin{array}{lcl}
\widetilde{K}_\varphi(x,y) &=& xK(\varphi(x,y)) = xy\left(t\displaystyle\sum_{(i,j)\in\mathcal{S}} d_{i,j}x^{i-j}y^{i} - 1\right) = xy\left(t\displaystyle\sum_{(i,j)\in\{0,\pm1\}^2} d_{j,j-i}x^{i}y^{j} - 1\right),\\
f(x) &=& td_{-1,0} + txd_{-1,-1},\\
g(x,y) &=& -y\Big(-\frac{1}{2} + t\Big(\frac{1}{2}(d_{1,1}y + d_{0,0} + d_{-1,-1}y^{-1}) + d_{0,-1}x + d_{1,0}xy\Big)\Big),\\
L_\varphi(x,y) &=& L(\varphi(x,y)) = \displaystyle\sum_{\substack{i\geqslant 1\\ j\geqslant 0\\ n\geqslant 0}} c_{j,j-i}(n)x^{i}y^{j}t^{n},\\
C_{0-}(x) &=& \displaystyle\sum_{\substack{i>0\\ n\geqslant 0}} c_{0,-i}(n)x^{i}t^{n},\\
D_\varphi(y) &=& D(\varphi(x,y)) = \displaystyle\sum_{\substack{i\geqslant 0\\ n\geqslant 0}} c_{i,i}(n)y^{i}t^{n}.
\end{array}
$$

Note that f,g are both polynomials. The multiplication by x in (11) permits us to find a kernel $\widetilde{K}_\varphi(x,y)$ which is on the same form as the original kernel $K(x,y)$. More precisely, the following lemma holds, and is a straigthforward consequence of the expression of $\widetilde{K}_\varphi(x,y)$.

Lemma 1.3. *The polynomial* $\widetilde{K}_\varphi(x,y)$ *corresponds to a kernel of a weighted walk with weight* $d^{\varphi}_{i,j}$*, where*

$$
\begin{array}{lcllcllcl}
d^{\varphi}_{-1,1} &=& 0, & d^{\varphi}_{0,1} &=& d_{1,1}, & d^{\varphi}_{1,1} &=& d_{1,0},\\
d^{\varphi}_{-1,0} &=& d_{0,1}, & d^{\varphi}_{0,0} &=& d_{0,0}, & d^{\varphi}_{1,0} &=& d_{0,-1},\\
d^{\varphi}_{-1,-1} &=& d_{-1,0}, & d^{\varphi}_{0,-1} &=& d_{-1,-1}, & d^{\varphi}_{1,-1} &=& 0.
\end{array}
$$

By Lemma 1.3, $\widetilde{K}_\varphi(x,y)$ has degree at most two in x and degree at most two in y. We may then write

$$
\widetilde{K}_\varphi(x,y) = \widetilde{a}_\varphi(y)x^2 + \widetilde{b}_\varphi(y)x + \widetilde{c}_\varphi(y) = a_\varphi(x)y^2 + b_\varphi(x)y + c_\varphi(x),
$$

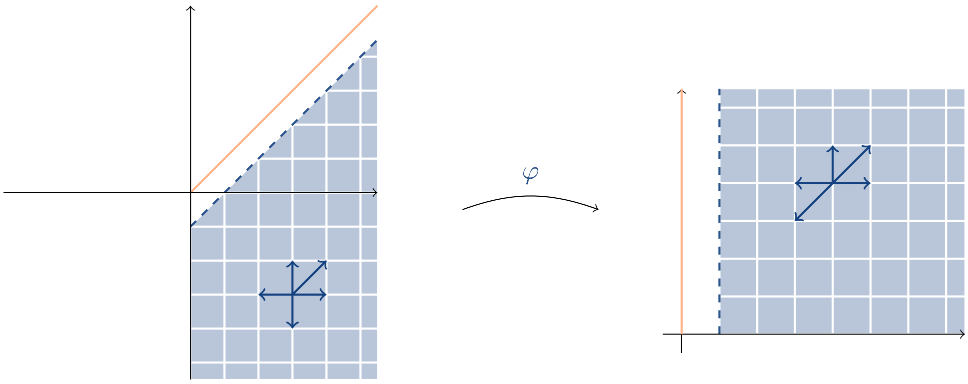


FIGURE 5. After performing the change of coordinates φ defined in (10), the lower octant is transformed into the right quadrant. The weights of the walk are also changed by φ, see Lemma 1.3.

where:

$$\begin{cases} a_\varphi(x) = tx \sum\limits_{i\in\{0,\pm1\}} d^\varphi_{i,1}x^i; \quad b_\varphi(x) = tx \sum\limits_{i\in\{0,\pm1\}} d^\varphi_{i,0}x^i - x; \quad c_\varphi(x) = tx \sum\limits_{i\in\{0,\pm1\}} d^\varphi_{i,-1}x^i; \\ \widetilde{a}_\varphi(y) = ty \sum\limits_{j\in\{0,\pm1\}} d^\varphi_{1,j}y^j; \quad \widetilde{b}_\varphi(y) = ty \sum\limits_{j\in\{0,\pm1\}} d^\varphi_{0,j}y^j - y; \quad \widetilde{c}_\varphi(y) = ty \sum\limits_{j\in\{0,\pm1\}} d^\varphi_{-1,j}y^j. \end{cases}$$

When $\widetilde{K}_\varphi(x,y)$ has exactly degree two in x and y, we also define the discriminants in x and y of the kernel $\widetilde{K}_\varphi(x,y)$:

$$d_\varphi(x) = b_\varphi(x)^2 - 4a_\varphi(x)c_\varphi(x) \qquad \text{and} \qquad \widetilde{d}_\varphi(y) = \widetilde{b}_\varphi(y)^2 - 4\widetilde{a}_\varphi(y)\widetilde{c}_\varphi(y).$$

We then have $\widetilde{K}_\varphi(x, Y_{\varphi,\pm}(x)) = 0$ (resp. $\widetilde{K}_\varphi(X_{\varphi,\pm}(y), y) = 0$) where

$$Y_{\varphi,\pm}(x) = \frac{-b_\varphi(x) \pm \sqrt{d_\varphi(x)}}{2a_\varphi(x)}, \qquad \text{and} \qquad X_{\varphi,\pm}(y) = \frac{-\widetilde{b}_\varphi(y) \pm \sqrt{\widetilde{d}_\varphi(y)}}{2\widetilde{a}_\varphi(y)}.$$

1.4. **The kernel curve.** Let us consider the algebraic curve

$$E = \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid K(x,y) = 0\}.$$

The latter being not compact, we need to embed it into the projective line. We first recall that $\mathbb{P}^1(\mathbb{C})$ denotes the complex projective line, which is the quotient of $\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}$ by the equivalence relation $\sim$ defined by

$$(x_0, x_1) \sim (x'_0, x'_1) \Leftrightarrow \exists \lambda \in \mathbb{C}^*, (x'_0, x'_1) = \lambda(x_0, x_1).$$

The equivalence class of $(x_0, x_1) \in \mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}$ is usually denoted by $[x_0 : x_1] \in \mathbb{P}^1(\mathbb{C})$. The map $x \mapsto [x : 1]$ embeds $\mathbb{C}$ inside $\mathbb{P}^1(\mathbb{C})$. The latter map is not surjective: its image is $\mathbb{P}^1(\mathbb{C}) \setminus \{[1 : 0]\}$; the missing point $[1 : 0]$ is usually denoted by ∞. Now, we embed E inside $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ via $(x,y) \mapsto ([x : 1], [y : 1])$. The kernel curve $\overline{E}$ is the closure of this embedding of E. In other words, $\overline{E}$ is the algebraic curve defined by

$$\overline{E} = \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0, x_1, y_0, y_1) = 0\}$$

where $\overline{K}(x_0, x_1, y_0, y_1)$ is the following bihomogeneous polynomial

$$\overline{K}(x_0, x_1, y_0, y_1) = x_1^2 y_1^2 K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}\right) = t\sum_{i,j=0}^{2} d_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j} - x_0 x_1 y_0 y_1. \tag{12}$$

The following proposition has been proved in [DHRS20a, Proposition 2.1 and Corollary 2.6], when t is transcendental and has been extended for a general $0 < t < 1$ in [DR19, Proposition 9].

Proposition 1.4. *The following facts are equivalent.*

(i) $\overline{E}$ *is an elliptic curve;*

(ii) *The set of authorized directions* $\mathcal{S}$ *is not included in any half space.*

From now on, we make the following additional assumption.

(H2) The set of authorized direction $\mathcal{S}$ is not included in any half space.

Let

$$\overline{K}_\varphi(x_0, x_1, y_0, y_1) = x_1^2 y_1^2 \widetilde{K}_\varphi\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}\right) = t\sum_{i,j=0}^{2} d^{\varphi}_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j} - x_0 x_1 y_0 y_1,$$

and define

$$\overline{E}_\varphi = \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}_\varphi(x_0, x_1, y_0, y_1) = 0\}.$$

Proposition 1.5. *Assume that Assumptions (H1) and (H2) hold. Then,* $\overline{E}_\varphi$ *is an elliptic curve.*

Proof. By Lemma 1.3 and the fact that Assumptions (H1), (H2) hold, the set of authorized directions of $\widetilde{K}_\varphi(x, y)$ is not included in any half space. This is now Proposition 1.4. □

1.5. **The Group of the walk.** Remind that we have seen in the proof of Proposition 1.5, that $\widetilde{K}_\varphi(x, y)$ has degree two in x and y and valuation 0 in x and y. Then, $a_\varphi(x), c_\varphi(x), \widetilde{a}_\varphi(y), \widetilde{c}_\varphi(y)$ are not identically zero.

Following [BMM10, Section 3], [KY15, Section 3] or [FIM17], we consider the involutive rational functions[1]

$$i_1, i_2 : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \dashrightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

given by

$$i_1(x, y) = \left(\frac{x_0}{x_1}, \frac{c_\varphi(\frac{x_0}{x_1})}{a_\varphi(\frac{x_0}{x_1})\frac{y_0}{y_1}}\right) \text{ and } i_2(x, y) = \left(\frac{\widetilde{c}_\varphi(\frac{y_0}{y_1})}{\widetilde{a}_\varphi(\frac{y_0}{y_1})\frac{x_0}{x_1}}, \frac{y_0}{y_1}\right).$$

Note that i_1, i_2 are "only" rational functions in the sense that they are *a priori* not defined when the denominators vanish.

For a fixed value of x, there are at most two possible values of y such that $(x, y) \in \overline{E}_\varphi$. The involution i_1 corresponds to interchanging these values. A similar interpretation can be given for i_2. Then the kernel curve $\overline{E}_\varphi$ is left invariant by the natural action of i_1, i_2.

[1] In what follows, we use the classical dashed arrow notion to denote rational maps; *a priori*, such functions may not be defined everywhere.

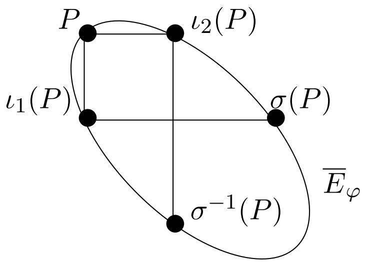


FIGURE 6. The maps i_1 and i_2 restricted to the kernel curve $\overline{E}_\varphi$

We denote by ι_1, ι_2 the restriction of i_1, i_2 on $\overline{E}_\varphi$, see Figure 6. Again, these functions are *a priori* not defined where the denominators vanish. However, by [DHRS20a, Proposition 3.1], who have been proved for t transcendental but stay valid for a general $0 < t < 1$, this is only an "apparent problem": ι_1 and ι_2 can be extended into morphisms of $\overline{E}_\varphi$. We recall that a rational map $f : \overline{E}_\varphi \dashrightarrow \overline{E}_\varphi$ is a morphism if it is regular at any $P \in \overline{E}_\varphi$, *i.e.* if f can be represented in suitable affine charts containing P and $f(P)$ by a rational function with nonvanishing denominator at P.

Let us finally define

$$\sigma = \iota_2 \circ \iota_1.$$

Note that such a map is known as a QRT-map and has been widely studied, see [Dui10]. In the quarter plane case, the algebraic nature of the generating series highly depends on the fact that σ has finite or infinite order. More precisely, in the unweighted quarter plane case, the group associated to the curve $\overline{E}$ is finite if and only if the generating series is D-finite. In the unweighted three quarter plane, when $\overline{E}$ is an elliptic curve, an infinite group associated to the curve $\overline{E}$ implies that the generating series is not D-finite, see [Mus19, Theorem 1.3]. Note that when the $d_{i,j}$ are fixed, the cardinality of the group depends upon t, see [FR11] for concrete examples.

Remind that $g(x, y)$ was defined on (11). We may now prove that on the curve $\overline{E}_\varphi$, $g(x, y)$ almost does not depends upon x. More precisely, the following lemma holds.

Lemma 1.6. *There exists $\varepsilon \in \{-1, 1\}$, such that if we evaluate $g(x, y)$ on $\overline{E}_\varphi$, then $g(x, y) = \frac{\varepsilon\sqrt{\widetilde{d}_\varphi(y)}}{2}$. Furthermore, for every $(x, y) \in \overline{E}_\varphi$,*

$$\iota_2(g(x, y)) = g(\iota_2(x), y) = -g(x, y).$$

Proof. Let $(x, y) \in \overline{E}_\varphi$. We find $x = \frac{-\widetilde{b}_\varphi(y)-\varepsilon\sqrt{\widetilde{d}_\varphi(y)}}{2\widetilde{a}_\varphi(y)}$ for some $\varepsilon \in \{-1, 1\}$. Since $g(x, y) = -\left(x\widetilde{a}_\varphi(y) + \frac{\widetilde{b}_\varphi(y)}{2}\right)$,

$$g(x, y) = -\frac{-\widetilde{b}_\varphi(y) - \varepsilon\sqrt{\widetilde{d}_\varphi(y)}}{2\widetilde{a}_\varphi(y)} \times \widetilde{a}_\varphi(y) - \frac{\widetilde{b}_\varphi(y)}{2} = \frac{\varepsilon\sqrt{\widetilde{d}_\varphi(y)}}{2}.$$

Furthermore, $\iota_2(x)=\frac{-\widetilde{b}_\varphi(y)+\varepsilon\sqrt{\widetilde{d}_\varphi(y)}}{2\widetilde{a}_\varphi(y)}$ is the other root, and the same computation shows

$$g(\iota_2(x),y)=-\frac{\varepsilon\sqrt{\widetilde{d}_\varphi(y)}}{2}=-g(x,y).$$

□

2. Analytic continuation

2.1. **Uniformization of the kernel curve.** Since by Proposition 1.5, $\overline{E}_\varphi$ is an elliptic curve, we may identify $\overline{E}_\varphi$ with $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$, with $(\omega_1,\omega_2)\in\mathbb{C}^2$ basis of a lattice, via the $(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$-periodic map

$$\begin{array}{llll}\Lambda: & \mathbb{C} & \to & \overline{E}_\varphi \\ & \omega & \mapsto & (\mathfrak{q}_1(\omega),\mathfrak{q}_2(\omega)),\end{array}$$

where $\mathfrak{q}_1,\mathfrak{q}_2$ are rational functions of $\wp$ and its derivative $d\wp/d\omega$, where $\wp$ is the Weierstrass function associated with the lattice $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$:

$$\wp(\omega)=\wp(\omega;\omega_1,\omega_2):=\frac{1}{\omega^2}+\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{1}{(\omega+\ell_1\omega_1+\ell_2\omega_2)^2}-\frac{1}{(\ell_1\omega_1+\ell_2\omega_2)^2}\right).$$

Then, the field of meromorphic functions on $\overline{E}_\varphi$ may be identified with the field of meromorphic functions on $\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$, i.e. the field of meromorphic functions on $\mathbb{C}$ that are (ω_1,ω_2)-periodic (or elliptic). Classically, this latter field is equal to $\mathbb{C}(\wp,\wp')$, see [WW96]. The map Λ induces a bijection from $\mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$ to $\overline{E}_\varphi$ we will still denote by Λ.

The maps ι_1, ι_2 and σ may be lifted to the ω-plane. We will call them $\widetilde{\iota}_1$, $\widetilde{\iota}_2$ and $\widetilde{\sigma}$, respectively. So we have the commutative diagrams

$$\begin{tikzcd} \overline{E}_\varphi \arrow[r, "\iota_k"] & \overline{E}_\varphi \\ \mathbb{C} \arrow[u, "\Lambda"] \arrow[r, "\widetilde{\iota}_k"'] & \mathbb{C} \arrow[u, "\Lambda"'] \end{tikzcd} \qquad \begin{tikzcd} \overline{E}_\varphi \arrow[r, "\sigma"] & \overline{E}_\varphi \\ \mathbb{C} \arrow[u, "\Lambda"] \arrow[r, "\widetilde{\sigma}"'] & \mathbb{C} \arrow[u, "\Lambda"'] \end{tikzcd}$$

More precisely, following [Dui10] (see in particular Proposition 2.5.2, Page 35 and Remark 2.3.8), and [DR19, Section 2.1], there exists $\omega_3\in\mathbb{C}$ such that

$$\widetilde{\iota}_1(\omega)=-\omega,\quad \widetilde{\iota}_2(\omega)=-\omega+\omega_3,\quad\text{and}\quad \widetilde{\tau}(\omega)=\omega+\omega_3. \tag{13}$$

We have $\Lambda(\omega)=\Lambda(\omega+\omega_1)=\Lambda(\omega+\omega_2)$ for all $\omega\in\mathbb{C}$. This implies that for all $\ell_1,\ell_2\in\mathbb{Z}$, $\widetilde{\iota}_2$ may be replaced by $\omega\mapsto-\omega+\omega_3+\ell_1\omega_1+\ell_2\omega_2$. This shows that ω_3 is not uniquely defined: it is only defined modulo the lattice $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$.

An explicit expression of $\omega_1,\omega_2,\omega_3$ and Λ may be found for instance in [DR19, Section 2]. It is proved that without loss of generalities, we may assume that ω_1 is purely imaginary, and ω_2,ω_3 are positive real numbers, with $0<\omega_3<\omega_2$.

2.2. **Analytic continuation.** Define the domains

$$\mathcal{D}_x := \overline{E}_\varphi \bigcap \{|x| < 1\}, \quad \mathcal{D}_y := \overline{E}_\varphi \bigcap \{|y| < 1\}, \quad \text{and} \quad \mathcal{D}_{x,y} := \mathcal{D}_x \bigcap \mathcal{D}_y. \tag{14}$$

The coefficients of the series C_{0-} are positive real numbers bounded by 1. Since $0 < t < 1$, we find that C_{0-} converges in $\mathcal{D}_x$. Therefore $xf(x)C_{0-}(x)$ converges in $\mathcal{D}_x$. Similarly, $D_\varphi(y)$ converges in $\mathcal{D}_y$, and $L_\varphi(x,y)$ converges in $\mathcal{D}_{x,y}$. We now follow the strategy of [DR19, Section 2]. Since the first steps are almost the same, we sketch the method. We evaluate (11) on the non-empty set $\mathcal{D}_{x,y}$ to find,

$$0 = xf(x)C_{0-}(x) + xg(x,y)D_\varphi(y) + \frac{t}{2}d_{-1,-1}xC_{0,0} - \frac{xy}{2}.$$

Let us prove that there are at most two points of $\overline{E}_\varphi$ with x-coordinate equal to $[0:1]$. Indeed, $([0:1],[y_0:y_1]) \in \overline{E}_\varphi$ implies that $[y_0:y_1]$ satisfies

$$0 = t\sum_{j=0}^{2} d^{\varphi}_{-1,j-1}y_0^j y_1^{2-j} = t(d^{\varphi}_{-1,0}y_0y_1 + d^{\varphi}_{-1,-1}y_1^2).$$

It follows that there are at most two points in $\overline{E}_\varphi$ of the form $([0:1],[y_0:y_1])$:

$$([0:1],[1:0]), \quad ([0:1],[-d^{\varphi}_{-1,-1}:d^{\varphi}_{-1,0}]).$$

Let $\mathcal{Z} \subset \overline{E}_\varphi$ denotes the finite set of points with x-coordinate that is equal to $[0:1]$. On $\mathcal{D}_{x,y} \setminus \mathcal{Z}$, we may simplify by x:

$$0 = f(x)C_{0-}(x) + g(x,y)D_\varphi(y) + \frac{t}{2}d_{-1,-1}C_{0,0} - \frac{y}{2}. \tag{15}$$

Since the right hand side is analytic on $\mathcal{D}_{x,y}$, it is continuous, and (15) holds on $\mathcal{D}_{x,y}$. We continue $g(x,y)D_\varphi(y)$ on $\mathcal{D}_x$ with the formula

$$g(x,y)D_\varphi(y) = -f(x)C_{0-}(x) - \frac{t}{2}d_{-1,-1}C_{0,0} + \frac{y}{2}.$$

Similarly, we extend $f(x)C_{0-}(x)$ on $\mathcal{D}_y$. So both functions has been extended to $\mathcal{D}_x \cup \mathcal{D}_y$. Let us now see their analytic continuation in the ω-plane. By [DR19, Section 2], there exists a connected set $\mathcal{O} \subset \mathbb{C}$ such that

- $\Lambda(\mathcal{O}) = \mathcal{D}_x \cup \mathcal{D}_y$;
- $\widetilde{\sigma}^{-1}(\mathcal{O}) \cap \mathcal{O} \neq \varnothing$;
- $\bigcup_{\ell\in\mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$.

The monomials x and y are meromorphic on $\overline{E}_\varphi$. We will denote them by $x(\omega)$ and $y(\omega)$ respectively. Note that we have explicit expression for the latter, see [DR19, Section 2]. Simplifying notations, let us write $g_{x,y}(\omega) = g(x(\omega), y(\omega))$.

The main difference with [DR19, Section 2] begin here, since in front of $D_\varphi(y)$, the polynomial $g(x,y)$ involves x and y. The use of Lemma 1.6 will be a crucial point in the proof of the following theorem.

Similarly to the quarter plane case, we have the following functional equations.

Theorem 2.1. *The functions $f(x)C_{0-}(x)$ and $g(x,y)D_{\varphi}(y)$ may be lifted to the universal cover of $\overline{E}_{\varphi}$. We will call respectively r_x and r_y the continuations. Seen as functions of ω, they are meromorphic on $\mathbb{C}$ and satisfy*

$$r_x(\omega+2\omega_3)=r_x(\omega)+y(\omega+2\omega_3)-y(\omega+\omega_3), \tag{16}$$

$$r_x(\omega+\omega_1)=r_x(\omega), \tag{17}$$

$$r_y(\omega+2\omega_3)=r_y(\omega)+\frac{-y(\omega)+2y(\omega+\omega_3)-y(\omega+2\omega_3)}{2}, \tag{18}$$

$$r_y(\omega+\omega_1)=r_y(\omega). \tag{19}$$

Proof. Let $\widetilde{r}_y$ be the meromorphic continuation of $D_{\varphi}(y)$. Consider $x(\omega)$ and $y(\omega)$ as meromorphic functions on $\overline{E}_{\varphi}$. From (15), we deduce that for all $\omega\in\mathcal{O}$,

$$0=r_x(\omega)+g_{x,y}(\omega)\widetilde{r}_y(\omega)+\frac{td_{-1,-1}C_{0,0}}{2}-\frac{1}{2}y(\omega). \tag{20}$$

Remind that $\widetilde{\iota}_1(\omega)=-\omega$ and $\iota_1(x)=x$, so that $x(\omega)=x(-\omega)$ and $r_x(\omega)=r_x(-\omega)$. Similarly, $\widetilde{\iota}_2(\omega)=-\omega+\omega_3$ and $\iota_2(y)=y$, so that $y(-\omega)=y(\omega+\omega_3)$ and $\widetilde{r}_y(-\omega)=\widetilde{r}_y(\omega+\omega_3)$.

By Lemma 1.6, $\widetilde{\sigma}(g_{x,y}(\omega))=-\widetilde{\iota}_1(g_{x,y}(\omega))$. We now apply $\widetilde{\iota}_1$ in both sides. We deduce that for all $\omega\in\widetilde{\sigma}^{-1}(\mathcal{O})\cap\mathcal{O}$,

$$0=r_x(\omega)-g_{x,y}(\omega+\omega_3)\widetilde{r}_y(\omega+\omega_3)+\frac{td_{-1,-1}C_{0,0}}{2}-\frac{y(\omega+\omega_3)}{2}. \tag{21}$$

We now apply $\widetilde{\iota}_2$ in both sides to we obtain for all $\omega\in\widetilde{\sigma}^{-1}(\mathcal{O})\cap\mathcal{O}$,

$$0=r_x(\omega+\omega_3)+g_{x,y}(\omega+\omega_3)\widetilde{r}_y(\omega+\omega_3)+\frac{td_{-1,-1}C_{0,0}}{2}-\frac{y(\omega+\omega_3)}{2}. \tag{22}$$

By definition, $r_y=g_{x,y}\widetilde{r}_y$. Subtracting (21) to (20), we find

$$r_y(\omega+\omega_3)=-r_y(\omega)+\frac{y(\omega)-y(\omega+\omega_3)}{2}. \tag{23}$$

Adding (21) to (22) we find

$$r_x(\omega+\omega_3)=-r_x(\omega)-td_{-1,-1}C_{0,0}+y(\omega+\omega_3). \tag{24}$$

Remind that $\widetilde{\sigma}^{-1}(\mathcal{O})\cap\mathcal{O}\neq\varnothing$, so that the intersection is an open set with an accumulation point. By the analytic continuation principle we may continue r_y and r_x on $\widetilde{\sigma}(\mathcal{O})$ with (23) and (24). Iterating this strategy, we continue r_x and r_y on $\bigcup_{\ell\in\mathbb{Z}}\widetilde{\sigma}^{\ell}(\mathcal{O})=\mathbb{C}$ and the functions satisfy (23) and (24). Now, let us iterate,

$$\begin{aligned} r_y(\omega+2\omega_3)&=-r_y(\omega+\omega_3)+\frac{y(\omega+\omega_3)-y(\omega+2\omega_3)}{2}\\ &=r_y(\omega)+\frac{-y(\omega)+2y(\omega+\omega_3)-y(\omega+2\omega_3)}{2}. \end{aligned}$$

Similarly, we find (16).

The proof of (17) and (19) is similar to [DR19]: it is based on the fact that r_x and r_y defined on $\mathcal{O}$ are already ω_1-periodic. □

3. Differential transcendence

3.1. **General criterion.** In what follows, every rings contain $\mathbb{Q}$. In particular every fields are of characteristic zero. A differential ring (L,δ) is a ring L equipped with a derivation δ, that is an additive morphism satisfying the Leibniz rules $\delta(ab)=\delta(a)b+a\delta(b)$ for every $a,b\in L$. If L is additionally a field, we say that it is a differential field. Let K be a differential field and let L be a K-algebra. Assume that the derivative of L extends the derivative of K. We say that $y\in L$ is δ-algebraic over K if there exists $n\in\mathbb{N}$ such that $y,\dots,\delta^n(y)$ are algebraically dependent over K. We say that y is δ-transcendental otherwise. When no confusions arise, we may also say that y is differentially algebraic or differentially transcendental.

We now want to embed the series $C(x,y)$ into a differential field. Since we consider walk with small steps, we have $c_{i,j}(n)=0$ when $|i|,|j|>n$. Therefore, $C(x,y)\in\mathcal{L}:=\mathbb{Q}(x,y)((t))$. Note that $\mathcal{L}$ is a field. It may be equipped as a differential field with the derivatives ∂_x and ∂_y. We also have $\mathbb{Q}\subset\mathbb{Q}(x)\subset\mathcal{L}$. Note that since x is ∂_x-algebraic over $\mathbb{Q}$, we find that $C(x,y)$ is ∂_x-algebraic over $\mathbb{Q}(x)$ if and only if it is ∂_x-algebraic over $\mathbb{Q}$. A similar statement holds for ∂_y.

Remark 3.1. *Let $f\in\mathcal{L}$. With the same reasons as in* [DH19, Remark 1.3]*, f is ∂_x-algebraic over $\mathbb{C}$ if and only if it is ∂_x-algebraic over $\mathbb{Q}$. A similar statement holds for ∂_y.*

On the other hand we will have to consider the differential transcendence of $r_x\in\mathcal{M}(\mathbb{C})$, where $\mathcal{M}(\mathbb{C})$ denotes the field of meromorphic function on $\mathbb{C}$. Note that both field $\mathbb{C}(\wp,\wp')\subset\mathcal{M}(\mathbb{C})$ are ∂_ω-fields. With the same reasons as above, r_x is ∂_ω-algebraic over $\mathbb{C}(\wp,\wp')$ if and only if it is ∂_ω-algebraic over $\mathbb{C}$. The following lemma shows that to determine the nature of $C(x,y)$, it then suffices to determine the nature of r_x.

Lemma 3.2. *The following statements are equivalent.*

(i) *$C(x,y)$ is ∂_x-algebraic over $\mathbb{Q}$;*
(ii) *$c(x)C_{-0}(x^{-1})$ is ∂_x-algebraic over $\mathbb{Q}$;*
(iii) *$C(x,y)$ is ∂_y-algebraic over $\mathbb{Q}$;*
(iv) *$\widetilde{c}(y)C_{0-}(y^{-1})$ is ∂_y-algebraic over $\mathbb{Q}$;*
(v) *$r_x(\omega)$ is ∂_ω-algebraic over $\mathbb{C}$.*

Proof. Note that $K(x,y)$ and $K(x,y)^{-1}$ are ∂_x-algebraic over $\mathbb{Q}$. Then $C(x,y)$ is ∂_x-algebraic over $\mathbb{Q}$ if and only if $K(x,y)C(x,y)$ is ∂_x-algebraic over $\mathbb{Q}$. By (7), $K(x,y)C(x,y)-c(x)C_{-0}(x^{-1})$ is a function of y and therefore is ∂_x-algebraic over $\mathbb{Q}$. This shows that (i) and (ii) are equivalent. Similarly, we prove that (iii) and (iv) are equivalent. By Assumption (H1), the weights $d_{i,j}$ are symmetric so $C(x,y)=C(y,x)$. This shows that (i) and (iii) are equivalent.

Note that (iv) is equivalent to the fact that $f(x)C_{0-}(x)\in\mathbb{Q}(x)((t))$ is differentially algebraic over $\mathbb{Q}$. By Remark 3.1, this is equivalent to the fact that $f(x)C_{0-}(x)$ is differentially algebraic over $\mathbb{C}$. The map Λ locally induces a bijection. By construction, we have

$$(fC_{0-})\circ\Lambda = r_x.$$

Since Λ involves the differentially algebraic functions $\wp,\wp'$, we find that Λ is differentially algebraic over $\mathbb{C}$. By [DHRS18, Lemmas 6.3 and 6.4], $f(x)C_{0-}(x)$ is differentially algebraic over $\mathbb{C}$ if and only if (v) holds. $\square$

By Theorem 2.1, r_x satisfies an equation of the form

$$\widetilde{\sigma}^2(r_x) = r_x + \widetilde{a},$$

where

$$\widetilde{a}(\omega) = y(\omega+2\omega_3) - y(\omega+\omega_3) \in \mathbb{C}(\wp,\wp').$$

From now on, we make the following assumption:

(H3) The automorphism σ has infinite order.

By Assumption (H3), $\widetilde{\sigma}$ has an infinite order. Such equation has been studied in a galoisian point of view in [HS08] and we have criteria on $\widetilde{a}$ to determine whether r_x is differentially algebraic or not. Note that the results of [DHRS18] cited in this section have been proved for t transcendental but stay valid for a general $0<t<1$. Combining [DHR18, Proposition 3.6] and [DHRS18, Proposition 6.2], we find:

Proposition 3.3. *The function r_x is ∂_ω-algebraic over $\mathbb{C}$ if and only if there exist $n\in\mathbb{N}$, $c_0,\dots,c_n\in\mathbb{C}$ with $c_n\neq 0$, and $h\in\mathbb{C}(\wp,\wp')$, such that*

$$\sum_{\ell=0}^{n} c_\ell \partial_\omega^\ell\left(\widetilde{a}\right) = \widetilde{\sigma}^2(\widetilde{h}) - \widetilde{h}. \tag{25}$$

We are now concerned to give applicable criterias based on Proposition 3.3 to prove the differential transcendence of the generating series.

The first one uses the $\widetilde{\sigma}^2$-orbits of the poles of $\widetilde{a}$. To simplify the computations, we are going to consider $\widetilde{a}$ as a meromorphic functions on $\overline{E}_\varphi$. Let $\mathbf{a} := \sigma^2(y)-\sigma(y)$, such that $\widetilde{a}=\mathbf{a}\circ\Lambda$. Remind, see Lemma 3.2, that $r_x(\omega)$ is ∂_ω-transcendental over $\mathbb{C}$, if and only if $C(x,y)$ is ∂_x-transcendental over $\mathbb{Q}$, if and only if $C(x,y)$ is ∂_y-transcendental over $\mathbb{Q}$.

Proposition 3.4 (Corollary 3.7, [DHRS18])**.** *Assume that $\mathbf{a}$ has a pole P of order $\geqslant m$, such that none of the $\sigma^{2\ell}(P)$, with $\ell\in\mathbb{Z}^*$, is a pole of order $\geqslant m$ of $\mathbf{a}$. Then, r_x is ∂_ω-transcendental over $\mathbb{C}$.*

When Proposition 3.4 does not apply, we need a more precise result that involve the residues of $\widetilde{a}$. Note that by (H3), $\widetilde{\sigma}^2$ has infinite order and since $\widetilde{a}$ has a finite number of poles modulo $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, we find that for all $\omega_0\in\mathbb{C}$, $\widetilde{a}$, has a finite number of poles in $\omega_0+2\mathbb{Z}\omega_3$.

Proposition 3.5 (Proposition B.2, [DHRS18])**.** *The function r_x is ∂_ω-algebraic over $\mathbb{C}$ if and only if for all $\omega_0\in\mathbb{C}$, the following function is analytic at ω_0:*

$$\sum_{i=1}^{t}\widetilde{a}(\omega+2n_i\omega_3)$$

where n_i are the integers such that $\omega_0+2n_i\omega_3$ are poles of $\widetilde{a}$.

So to determine the nature of r_x, it suffices to:

- Compute the poles of $\mathbf{a}$ and their order.
- Compute the σ^2-orbits of the poles of $\mathbf{a}$.
- Compute the residues at the different poles.

The first step is solved by the following lemma.

Lemma 3.6. *The following holds:*

- *If $d^{\varphi}_{0,1} = 0$, then the poles of $\mathbf{a}$ are double and equal to*
$$\sigma^{-1}([0:1],[1:0]), \quad \sigma^{-2}([0:1],[1:0]).$$
- *If $d^{\varphi}_{0,1} \neq 0$, then the poles of $\mathbf{a}$ are simple and equal to*
$$\sigma^{-1}(P_1), \sigma^{-1}(P_2), \sigma^{-2}(P_1), \sigma^{-1}(P_2),$$
where
$$P_1 = ([0:1],[1:0]) \text{ and } P_2 = ([-d^{\varphi}_{0,1} : d^{\varphi}_{1,1}],[1:0]).$$

Proof. Let us begin by computing the poles of y and their multiplicities. It suffices to solve
$$\overline{K}_{\varphi}(x_0,x_1,1,0) = d^{\varphi}_{0,1}x_0x_1 + d^{\varphi}_{1,1}x_0^2 = 0.$$
We then find that

- if $d^{\varphi}_{0,1} = 0$, then y has a double pole $([0:1],[1:0])$.
- if $d^{\varphi}_{0,1} \neq 0$, then y has two simple poles, $([-d^{\varphi}_{0,1} : d^{\varphi}_{1,1}],[1:0])$, and $([0:1],[1:0])$.

We conclude by noticing that P is a pole of y if and only if $\sigma^{-n}(P)$ is a pole of $\sigma^n(y)$. □

Let us now focus on the second problem. We define an equivalence relation on $\overline{E}_{\varphi}$ as follows. We say that $A, B \in \overline{E}_{\varphi}$ satisfies $A \sim B$ if and only if there exists $n \in \mathbb{Z}$, such that $\sigma^{2n}(A) = B$. Then, given A, B, poles of $\mathbf{a}$, we want to determine whether $A \sim B$ or not.

A first partial result is the following.

Lemma 3.7. *Let $P \in \overline{E}_{\varphi}$. Then, $P \not\sim \sigma(P)$.*

Proof. To the contrary, assume the existence of $n \in \mathbb{Z}$ such that $\sigma^{2n}(P) = \sigma(P)$. Then, $\sigma^{2n-1}(P) = P$. Since $\widetilde{\sigma}(\omega) = \omega + \omega_3$, we find that σ fixes one point if and only if it is the identity. It is obviously not the case, leading to a contradiction. □

We now want to consider the ∂_t-transcendence so in the following paragraph, t is not fixed anymore and is considered as a variable. When we will consider the generating series as a function of t, we will denote it by $C(x,y;t)$. The field $\mathcal{L} = \mathbb{Q}(x,y)((t))$ is also a ∂_t-field. We have $\mathbb{Q} \subset \mathbb{Q}(x,y,t) \subset \mathcal{L}$, and $C(x,y;t)$ is ∂_x-algebraic over $\mathbb{Q}(x,y,t)$ if and only if it is ∂_x-algebraic over $\mathbb{Q}$. A similar statement holds for ∂_y and for ∂_t. We say that $C(x,y;t)$ is D-algebraic if and only if it is both ∂_x, ∂_y and ∂_t-algebraic over $\mathbb{Q}$. We say that $C(x,y;t)$ is D-transcendental otherwise. From what precede, this corresponds to the definition given in the introduction. The link with ∂_x-transcendence of $C(x,y;t)$ and $x \mapsto C(x,y)$ for t transcendental fixed is now made in the following lemma.

Lemma 3.8. *The following facts are equivalent.*

- *The series $C(x,y;t)$ is ∂_x-algebraic over $\mathbb{Q}$.*
- *For all $0<t<1$ fixed, $x\mapsto C(x,y)$ is ∂_x-algebraic over $\mathbb{Q}$.*

A similar statement holds for ∂_y.

Proof. If the first item holds, it is clear that the second item holds.

Let us prove the converse. Assume that for all $0<t<1$ fixed, $x\mapsto C(x,y)$ is ∂_x-algebraic over $\mathbb{Q}$. For $0<t<1$, let $\mathcal{L}_t$ be a differential equations such that $\mathcal{L}_t(C(x,y))$. Let us consider a transcendental number $t'\in]0,1[$. Since t' is transcendental and $C(x,y;t)\in\mathbb{Q}(x,y)((t))$, we obtain that $\mathcal{L}'_t(C(x,y;t))=0$. This concludes the proof for the ∂_x-derivation. The same reasoning holds for ∂_y. □

Theorem 3.9. *Assume that Assumptions (H1), (H2), and (H3) hold. Assume further that $d_{1,1}=d^{\varphi}_{0,1}=0$. Then, $C(x,y;t)$ is D-transcendental. Let us fix $0<t<1$. More precisely, $C(x,y)$ is ∂_x-transcendental over $\mathbb{Q}$, and ∂_y-transcendental over $\mathbb{Q}$.*

Proof. By Lemma 3.6, **a** has two double poles, $\sigma^{-1}([0:1],[1:0])$, $\sigma^{-2}([0:1],[1:0])$. By Lemma 3.7 they do not belong to the same σ^2-orbits. By Proposition 3.4 and Lemma 3.2, $C(x,y)$ is ∂_x-transcendental over $\mathbb{Q}$, and ∂_y-transcendental over $\mathbb{Q}$. By Lemma 3.8, $C(x,y;t)$ is ∂_x-transcendental over $\mathbb{Q}$, and by definition, $C(x,y;t)$ is D-transcendental. □

When $d_{1,1}=d^{\varphi}_{0,1}\neq 0$, the situation is more complicated. We have four simple poles $\sigma^{-1}(P_1),\sigma^{-1}(P_2),\sigma^{-2}(P_1),\sigma^{-1}(P_2)$ and we need to determine the σ^2-orbits. By Lemma 3.7, we know that $\sigma^{-1}(P_1)\not\sim\sigma^{-2}(P_1)$, and $\sigma^{-1}(P_2)\not\sim\sigma^{-2}(P_2)$. To compute completely the σ^2-orbits, we need to determine whether $P_1\sim P_2$, (and then $\sigma^{\ell}(P_1)\sim\sigma^{\ell}(P_2)$ for every $\ell\in\mathbb{Z}$), and whether $P_1\sim\sigma(P_2)$ (and then $\sigma^{\ell}(P_1)\sim\sigma^{\ell\pm1}(P_2)$ for every $\ell\in\mathbb{Z}$). We will see that in practice, it suffices to determine whether $P_1\sim P_2$ or not.

Let us now consider the third problem. By Theorem 3.9, we may reduce to the case where $d_{1,1}=d^{\varphi}_{0,1}\neq 0$. If we see y as a meromorphic function in the ω-plane, we see that, modulo the lattice $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, is has two simple poles corresponding to P_1 and P_2. Let us see Λ as a bijection from $\mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$ to $\overline{E}_{\varphi}$. Let $[\omega_{y,1}],[\omega_{y,2}]\in\mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$ such that $\Lambda([\omega_{y,i}])=P_i$. Since y is an elliptic function, the sum of the residues in a fundamental parallelogram is zero. This shows that if $0\neq\alpha$ is the residue at $[\omega_{y,1}]$, then the residue at $[\omega_{y,2}]$ is $-\alpha$. Furthermore, it is clear that the residue at ω_0 of $f(\omega)\in\mathcal{M}er(\mathbb{C})$ equals to the residue at $\omega_0-\omega_3$ of $\widetilde{\sigma}(f(\omega))$. Then, the residues of the simple poles of $\widetilde{a}(\omega)=y(\omega+2\omega_3)-y(\omega+\omega_3)$ are equal to

$$([\omega_{y,1}]-\omega_3,-\alpha),\quad([\omega_{y,1}]-2\omega_3,\alpha),\quad([\omega_{y,2}]-\omega_3,\alpha),\quad([\omega_{y,2}]-2\omega_3,-\alpha).$$

Remind, see Lemma 3.7, that $P_i\not\sim\sigma(P_i)$. From what precede, the conclusion of Proposition 3.5 holds positively if and only if $P_1\sim P_2$ (and then $\sigma(P_1)\sim\sigma(P_2)$). We have proved:

Proposition 3.10. *Let us fix $0<t<1$. Assume that Assumptions (H1), (H2), and (H3) hold. Assume further that $d_{1,1}=d^{\varphi}_{0,1}\neq 0$. Then, $C(x,y)$ is ∂_x-algebraic over $\mathbb{Q}$ (resp. ∂_y-algebraic over $\mathbb{Q}$) if and only if*

$$P_1\sim P_2.$$

We are in fact able to prove the ∂_t-algebraicity, and then show that the generating series is D-algebraic.

Corollary 3.11. *Assume that Assumptions (H1), (H2), and (H3) hold. Assume further that* $d_{1,1} = d^{\varphi}_{0,1} \neq 0$ *and that there exists* $k \in \mathbb{Z}$*, such that for all* $0 < t < 1$*,* $P_1 = \sigma^{2k}(P_2)$*. Then,* $C(x, y; t)$ *is D-algebraic.*

Proof. By Proposition 3.10, for $0 < t < 1$ fixed, $x \mapsto C(x, y)$ is ∂_x-algebraic over $\mathbb{Q}$. By Lemma 3.8, $C(x, y; t)$ is ∂_x-algebraic over $\mathbb{Q}$. A similar statement holds for ∂_y. It remains to consider the t-derivation. Simplifying notations, let us drop the dependence in t in the functions.

By Lemma 3.6, the poles of $\mathbf{a}$ are simple and equal to $\sigma^{-1}(P_1), \sigma^{-1}(P_2), \sigma^{-2}(P_1), \sigma^{-1}(P_2)$. Furthermore, $P_1 \neq P_2$, and then $k \neq 0$. Up to interchanging P_1 and P_2, we may reduce to the case where $k > 0$.

Remind that $\alpha \neq 0$ is the residue of $y(\omega)$ at $[\omega_{y,1}]$, and $-\alpha$ is the residue of $y(\omega)$ at $[\omega_{y,2}]$. Let $\omega_{y,1} \in \mathbb{C}$ be a representative of $[\omega_{y,1}]$. Let $f_0(\omega) = \frac{1}{\wp(\omega-\omega_{y,1}+3\omega_3/2)-\wp(\omega_3/2)}$. It is an elliptic function with two simple poles at $[\omega_{y,1}] - \omega_3$, and $[\omega_{y,1}] - 2\omega_3$. There exists $c \in \mathbb{C}^*$, such that $f(\omega) = cf_0(\omega)$ is an elliptic function with two simple poles at $[\omega_{y,1}]-\omega_3$, and $[\omega_{y,1}] - 2\omega_3$ with respective residues α and $-\alpha$. Let $g(\omega) = \sum_{\ell=0}^{k-1} f(\omega + 2\ell\omega_3)$. Then, $\widetilde{\sigma}^2(g(\omega)) - g(\omega)$ has only simple poles with corresponding residues

$$([\omega_{y,1}] - (2k + 2)\omega_3, -\alpha), \quad ([\omega_{y,1}] - (2k + 1)\omega_3, \alpha), \quad ([\omega_{y,1}] - \omega_3, -\alpha), \quad ([\omega_{y,1}] - 2\omega_3, \alpha).$$

On the other hand, remind that the poles of $\widetilde{a}(\omega)$ are simple and equal to

$$([\omega_{y,1}] - \omega_3, -\alpha), \quad ([\omega_{y,1}] - 2\omega_3, \alpha), \quad ([\omega_{y,2}] - \omega_3, \alpha), \quad ([\omega_{y,2}] - 2\omega_3, -\alpha).$$

We now use $P_1 = \sigma^{2k}(P_2)$ to deduce that the latter pair of poles and residues are equal to

$$([\omega_{y,1}] - \omega_3, -\alpha), \quad ([\omega_{y,1}] - 2\omega_3, \alpha), \quad ([\omega_{y,1}] - (2k + 1)\omega_3, \alpha), \quad ([\omega_{y,1}] - (2k + 2)\omega_3, -\alpha).$$

Then, $\widetilde{\sigma}^2(g(\omega)) - g(\omega)$ and $\widetilde{a}(\omega)$ have same poles and residues. This shows that $\widetilde{\sigma}^2(g(\omega)) - g(\omega) - \widetilde{a}(\omega)$ is an elliptic function with no poles, it is therefore constant. Remind, see Theorem 2.1, that $\widetilde{\sigma}^2(r_x(\omega)) - r_x(\omega) = \widetilde{a}(\omega)$. Then, $g(\omega) - r_x(\omega)$ is $\widetilde{\sigma}^2$-invariant, i.e. it is $2\omega_3$-periodic. Since $g(\omega), r_x(\omega)$ are ω_1-periodic, see Theorem 2.1, $g(\omega)-r_x(\omega) \in \mathbb{C}(\wp_{1,3}, \wp'_{1,3})$, where $\wp_{1,3}$ denotes the Weierstrass function with respect to the periods $\omega_1, 2\omega_3$. This shows that $r_x(\omega) \in \mathbb{C}(\wp, \wp', \wp_{1,3}, \wp'_{1,3})$. By [BBMR17, Proposition 31 and Lemma 35], $r_x(\omega)$ and the map Λ are ∂_t-algebraic over $\mathbb{C}$. As we have seen in the proof of Lemma 3.2, $(fC_{0-}) \circ \Lambda = r_x$ and by [DHRS18, Lemmas 6.3 and 6.4], fC_{0-} is ∂_t-algebraic over $\mathbb{C}$. Then, $C_{0-}(y^{-1})$ is ∂_t-algebraic over $\mathbb{C}$. Since the weights are symmetric, $C(x, y) = C(y, x)$ and $C_{-0}(x^{-1})$ is also ∂_t-algebraic over $\mathbb{C}$. By [DH19, Remark 1.3], $C_{-0}(x^{-1})$ and $C_{0-}(y^{-1})$ are ∂_t-algebraic over $\mathbb{Q}$. By (7), $K(x, y)C(x, y)$ is ∂_t-algebraic over $\mathbb{Q}$. Then, $C(x, y)$ is ∂_t-algebraic over $\mathbb{Q}$. This concludes the proof. □

3.2. **Unweighted cases.** Let us now look at the unweighted symmetric models with infinite group:

After performing the change of variable φ, we obtain kernels corresponding to the following weights:

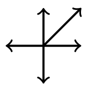
When $d_{0,1}^{\varphi} = 0$ we may use Theorem 3.9 to deduce the differential transcendence in the third case. Let us assume that $d_{0,1}^{\varphi} \neq 0$. We want to apply Proposition 3.10 and Corollary 3.11. Hopefully, in this situation, the σ-orbits (and therefore the σ^2-orbits) have been computed in [DHRS18, Section 6].

- In the first case, we find two distinct σ^2-orbits
$$\sigma^{-1}(P_1) \sim \sigma^{-2}(P_2), \quad \sigma^{-2}(P_1) \sim \sigma^{-1}(P_2).$$
- In the second case, we find two distinct σ^2-orbits
$$\sigma^{-1}(P_1) \sim \sigma^{-2}(P_2), \quad \sigma^{-2}(P_1) \sim \sigma^{-1}(P_2).$$
- In the fourth case, $\sigma^4(P_1) = P_2$ for all $0 < t < 1$, and then we find two distinct σ^2-orbits
$$\sigma^{-1}(P_1) \sim \sigma^{-1}(P_2), \quad \sigma^{-2}(P_1) \sim \sigma^{-2}(P_2).$$

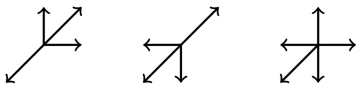
By Theorem 3.9, Proposition 3.10 and Corollary 3.11, we find:

Theorem 3.12. *The following holds:*

- *Assume that the model of the walk is the following:*



Then, $C(x, y; t)$ is D-algebraic.
- *Assume that the model of the walk is one of the following*



The series $C(x, y; t)$ is D-transcendental. Let us fix $0 < t < 1$. More precisely, $C(x, y)$ is ∂_x-transcendental over $\mathbb{Q}$. The same holds for ∂_y.

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