REFLECTED RANDOM WALKS AND UNSTABLE MARTIN BOUNDARY

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Abstract. We introduce a family of two-dimensional reflected random walks in the positive quadrant and study their Martin boundary. While the minimal boundary is systematically equal to a union of two points, the full Martin boundary exhibits an instability phenomenon, in the following sense: if some parameter associated to the model is rational (resp. non-rational), then the Martin boundary is discrete, homeomorphic to $\mathbb{Z}$ (resp. continuous, homeomorphic to $\mathbb{R}$). Such instability phenomena are very rare in the literature. Along the way of proving this result, we obtain several precise estimates for the Green functions of reflected random walks with escape probabilities along the boundary axes and an arbitrarily large number of inhomogeneity domains. Our methods mix probabilistic techniques and an analytic approach for random walks with large jumps in dimension two.

1. Introduction

1.1. **Martin boundary and instability.** Before formulating our main results, we recall the definition of Martin boundary and some associated key results in this field.

A brief account of Martin boundary theory. First introduced by Martin [31] for Brownian motion, the concept of Martin compactification is defined for countable Markov chains by Doob [11]. Consider a transient, irreducible, sub-stochastic Markov chain $Z = \{Z(n)\}_{n\geqslant 0}$ on a state space $E \subset \mathbb{Z}^d$, $d \geqslant 1$, with transition probabilities $\{p(x,y)\}_{x,y\in E}$. Given $x, y \in E$, the associated Green function $g(x,y)$ and Martin kernel $k(x,y)$ are respectively defined by

$$g(x,y) = \sum_{n=0}^{\infty} \mathbb{P}_x\big(Z(n) = y\big) \qquad \text{and} \qquad k(x,y) = \frac{g(x,y)}{g(x_0,y)},$$

where $\mathbb{P}_x$ denotes the probability measure on the set of trajectories of Z corresponding to the initial state $Z(0) = x$, and x_0 is a given reference point in E.

For irreducible Markov chains, the family of functions $\{k(\cdot,y)\}_{y\in E}$ is relatively compact with respect to the topology of pointwise convergence; in other words, for any sequence $\{y_n\}$ of points in E, there exists a subsequence $\{y_{n_k}\}$ along which the sequence $k(\cdot, y_{n_k})$ converges pointwise on E. The Martin compactification E_M is defined as the (unique) smallest compactification of E such that the Martin kernels $k(x,\cdot)$ extend continuously; a

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sequence $\{y_n\}_{n\geqslant 0}$ converges to a point of the Martin boundary $\partial_M E = E_M \setminus E$ of E if it leaves every finite subset of E and if the sequence of functions $k(\cdot, y_n)$ converges pointwise.

Recall that a function $h : E \to \mathbb{R}$ is harmonic for Z if, for all $x \in E$, $\mathbb{E}_x\big(h(Z(1))\big) = h(x)$. By the Poisson-Martin representation theorem, for every non-negative harmonic function h, there exists a positive Borel measure ν on $\partial_M E$ such that

$$h(x) = \int_{\partial_M E} k\big(x, \eta\big) d\nu(\eta).$$

By convergence theorem, for any $x \in E$, the sequence $\{Z(n)\}_{n\geqslant 0}$ converges $\mathbb{P}_x$-almost surely to a $\partial_M E$-valued random variable. The Martin boundary therefore provides all non-negative harmonic functions and shows how the Markov chain Z goes to infinity.

In order to identify the Martin boundary, one has to investigate all possible limits of the Martin kernel $k\big(x, y_n\big)$ when $|y_n| \to \infty$. As a consequence, the identification of the Martin boundary often reduces to the asymptotic computation of the Martin kernel or of the Green function. Such results are now well established for spatially homogeneous random walks, see, e.g., Ney and Spitzer [32] for $E = \mathbb{Z}^d$, see also the book of Woess [35] for many relevant references. On the other hand, as we shall see later in this introduction, the case of inhomogeneous random walks is much more involved and still largely open.

Instability of the Martin boundary. Once the notion of Martin boundary has been settled, it is natural to ask how stable it is with respect to the parameters of the model. Roughly speaking, throughout this paper, the Martin boundary will be called stable if the Martin compactification does not depend on small modifications of the transition probabilities.

Although we shall not use it in the present paper, let us recall the historically first way to measure the stability of the Martin boundary, which has been introduced by Picardello and Woess [34]. Define the ρ-Green function by

$$g\big(x, y; \rho\big) = \sum_{n=0}^{\infty} \mathbb{P}_x\big(Z(n) = y\big)\rho^{-n},$$

for ρ in the spectral interval. Then one may define the associated ρ-Martin kernel and ρ-Martin compactification. According to [34, Def. 2.4], the Martin boundary is stable if the ρ-Martin compactification does not depend on the eigenvalue ρ (with a possible exception at the critical value) and if the Martin kernels are jointly continuous w.r.t. space variable and eigenvalue.

The immense majority of known examples of Martin boundary is stable, see for example [33]. More precisely, to the best of our knowledge, the only known Markov process with unstable Martin boundary is a model of reflected random walk [20], as worked out by the first author of this paper.

Contribution. Our main objective is to shine a light on a rare instability phenomenon of the Martin boundary, in the framework of two-dimensional reflected random walks. More specifically, we propose a family of probabilistic models for which the arithmetic nature (rational vs. non-rational) of some parameter has a strong influence on the topology of the Martin boundary. Precisely, defining the (real) parameter t_0 as in (11), the Martin boundary will be homeomorphic to $\mathbb{Z}$ (resp. $\mathbb{R}$) if $t_0 \in \mathbb{Q}$ (resp. $t_0 \notin \mathbb{Q}$). As the quantity

t_0 is locally analytic in the transition probabilities (viewed as variables), one immediately deduces that the Martin boundary is unstable (in our definition). See Theorem 3 for a precise statement. We would like to emphasize the following two differences w.r.t. the only known other example of instability:

- The instability in [20] follows from an interplay between boundary and interior parameters. Our example only concerns interior parameters and is, from that point of view, more intrinsic.
- Being analytic and non-constant, the function t_0 in (11) takes infinitely many rational and non-rational values, and so the Martin boundary jumps infinitely many times from $\mathbb{Z}$ to $\mathbb{R}$. On the other hand, in the example found in [20], there is somehow only one jump, meaning that the spectral interval my be divided into two subintervals, and within each of them the Martin boundary is stable.

Interestingly, the quantity t_0 (whose arithmetic nature intervenes) appears in two recent papers [24, 17]; we will develop this point later on.

1.2. **Multidimensional reflected random walks in complexes.** This section is split into three parts. We shall first introduce the class of reflected random walks in the positive quadrant, on which we shall work throughout the paper and prove the instability phenomenon described above. We will next put this family of models into a larger class of multidimensional reflected random walks in complexes, in relation with many probabilistic questions in the last fifty years. Finally, we will review the literature from the viewpoint of Martin boundary and Green functions of reflected random walks (and more generally of non-homogeneous Markov chains).

Our model. Without entering into full details (more are to come in Section 2, where the model will be carefully introduced), we now define the main discrete stochastic process studied throughout this paper: it is a random walk in dimension two, reflected at the boundary of the quarter plane, with a finite (but arbitrarily large) number of homogeneity domains, see Figure 2. These domains are either points, half-lines, or a (translated) quadrant. Within each homogeneity domain, the model admits (spatially homogeneous) transition probabilities. Jumps into positive (North-East) directions may be unbounded, while we will place some restrictions on the size of the negative (South-West) jumps. See Assumptions 1, 2 and 3 in Section 2 for more details. We shall fix the parameters so as to have a transient Markov process, with escape probabilities along the axes, see Figure 1.

A general model of reflected random walks. Our probabilistic model belongs to a more general class of piecewise homogeneous models in multidimensional domains (the domains being usually simple cones, such as half-spaces or orthants, or union of cones, called complexes). This class of models is of great interest, as it is much richer than its fully homogeneous analogue, but still admits structured inhomogeneities, opening the way for a detailed analysis. Reflected random walks in the half-line (in dimension 1) are particularly studied in the literature, and so is the two-dimensional model of random walks in the quarter plane; for historical references, see Malyshev [28, 29, 30], Fayolle and Iasnogorodski [13], see also [14], Cohen and Boxma [7].

Many probabilistic features of these models are investigated in the literature, for instance in relation with the classification of Markov chains (recurrence and transience), see [15]. Another strong motivation to their study comes from their links with queueing systems [7, 8]. Furthermore, these models offer the opportunity to develop remarkable tools, e.g., complex analysis [7, 8, 25, 14] and asymptotic analysis [25]. Many combinatorial objects (maps, permutations, trees, Young tableaux, partitions, walks in Weyl chambers, queues, etc.) can be encoded by lattice walks, see [4] and references therein, so that understanding the latter, we will better understand the first objects. Let us also mention the article [10] by Denisov and Wachtel, which contains several fine estimates of exit times and local probabilities in cones. Finally, as it will be clear in the next paragraph, many interesting questions related to potential theory arise in the study of these non-homogeneous random walk models.

Martin boundary theory for non-homogeneous Markov chains. For such processes, the problem of explicitly describing the Martin compactification, or the Green functions asymptotics, is usually highly complex, and only few results are available in the literature. For random walks on non-homogeneous trees, the Martin boundary is described by Cartier [6]. Alili and Doney [1] identify the Martin boundary for a space-time random walk $S(n) = (Z(n), n)$ for a homogeneous random walk Z on $\mathbb{Z}$ killed when hitting the negative half-line. Doob [11] identifies the Martin boundary for Brownian motion on a half-space, by using an explicit form of the Green function. All these results are obtained by using the one-dimensional structure of the process.

In dimension two, a complex analysis method (based on the study of elliptic curves) is proposed by Kurkova and Malyshev [25] to identify and classify the Martin boundary for reflected nearest neighbor random walks with drift in $\mathbb{N}\times\mathbb{Z}$ and $\mathbb{N}^2$ ($\mathbb{N}$ denoting the set of non-negative integers $\{0, 1, 2, \dots\}$). In particular, although not put forward explicitly as such, [25, Thm. 2.6] contains an instability result of the Martin boundary (similar to the one we prove in our paper). This analytic approach was actually initially introduced by Malyshev [28, 29, 30] in his study of stationary distributions of ergodic random walks in the quarter plane. Let us also mention the work [26], where the second and third authors of the present article compute the Martin boundary of killed random walks in the quadrant, developing Malyshev's approach to that context. We emphasize that in [25, 26], exact asymptotics of the Green function (not only of the Martin kernel) are derived.

In order to identify the Martin boundary of a piecewise homogeneous (killed or reflected) random walk on a half-space $\mathbb{Z}^d \times \mathbb{N}$, a large deviation approach combined with Choquet-Deny theory and the ratio limit theorem of Markov-additive processes is proposed by Ignatiouk-Robert [19, 21, 22], and Ignatiouk-Robert and Loree [23]. It should be mentioned that some key arguments in [19, 21, 23, 22] are valid only for Markov-additive processes, meaning that the transition probabilities are invariant w.r.t. translations in some directions. In the previously cited articles, no exact asymptotics for Green functions are derived: only the Martin kernel asymptotics is considered. Building on the estimates of the local probabilities in cones derived in [10], the paper [12] obtains the asymptotics of the Green functions in this context.

1.3. Advances in the analytic and probabilistic approaches of random walks with large jumps.

Progress on the analytical method... We shall introduce the generating functions of the Green functions and prove that they satisfy various functional equations, starting from which we will deduce contour integral formulas for the Green functions. Applying asymptotic techniques to these integrals will finally lead to our main results.

Historically, the first techniques developed to study the above-mentioned problems were analytic, see in particular the pioneered works by Malyshev [28, 29, 30], Fayolle and Iasnogorodski [13]. In brief, the main idea consists in working on a Riemann surface naturally associated with the random walk, via the transition probability generating function. In turn, the fine study of the Riemann surface allows to observe and prove various probabilistic behaviors of the model.

This Riemann surface has genus one in the case of random walks with jumps to the eight nearest neighbors (sometimes called small step random walks). Such Riemann surfaces may be fully studied (e.g., using their parametrization with elliptic functions). This is the case considered in the early works [28, 29, 30, 13] as well as in subsequent papers on different models such as [25] or [27].

Larger jumps lead to higher genus Riemann surfaces, which become much harder to fully analyze, not to say impossible in general, as explained in the note [16]. Let us here mention the (combinatorial) contribution [2] in dimension one, and [16, 3] in dimension two, where some particular cases are studied. In the framework of random walks in the quarter plane with arbitrary big jumps, the books [7, 8] propose a theoretical study of relevant functional equations, concluding with the same difficulty that in general, one cannot expect a sufficiently precise study of the associated Riemann surface to deduce really explicit results (for instance on the stationary probabilities or Green functions).

For similar reasons, studying reflected or killed random walks in dimension three seems highly challenging, because of the need of describing the associated algebraic curve.

The main progress we do here is that we are able to dispense with the study of the Riemann surface in its entirety, and to replace it by a local study at only two points, which eventually concentrate all the information (at least from our asymptotic point of view). This local study being not at all sensitive to the size of the jumps (or to the genus of the surface), we are able to treat very general random walks in dimension two.

Our paper is therefore the first step in solving similar problems for random walks with big jumps in dimension two (for instance, join-the-shortest queue like problems) and in higher dimensions as well, in which the complete description of the algebraic curve is not possible in general.

We also emphasize that the number of homogeneity domains is arbitrarily large.

...using probabilistic techniques. In order to perform this reduction of the whole Riemann surface to two points, we combine the analytic method with several probabilistic estimates. More precisely, we will introduce simpler models, such as reflected random walks in half-spaces, and we will compare the Green functions of these models to those of our main random walk. In particular, we will prove that asymptotically, the main Green function

appears a sum of two terms, each of them can be interpreted in the light of half-space Green functions and further quantities related to one-dimensional random walks.

The combination between analytical and probabilistic aspects is visible on our results: we derive Theorem 1 by probabilistic methods, then Theorem 2 by analytical methods; the union of the two results gives our main Theorem 3 on the instability of the Martin boundary.

Related research. To conclude this introduction, we highlight future projects in relation with the present work. First, the progress we did in the understanding of the techniques could be applied to various other cases of two-dimensional reflected random walks. In this paper, we choose to focus on a model with escape probabilities along the two axes, because of our initial motivation related to the instability phenomenon. As a second extension of our techniques and results, we would like to look at higher dimensional reflected random walk models. A third project is to study the precise link between our definition of stability and that of Picardello and Woess [33].

Structure of the paper.

- Section 2: presentation of the main model and related models
- Section 3: statements of the main results (Theorems 1, 2 and 3)
- Section 4: rough uniform estimates of the Green functions (Proposition 3)
- Section 5: preliminary results to the proof of Theorem 1
- Section 6: proof of Theorem 1
- Section 7: preliminary results to the proof of Theorems 2 and 3
- Section 8: proof of Theorems 2 and 3
- Appendix A: proof of Proposition 2
- Appendix B: glossary of the hitting times used throughout the paper

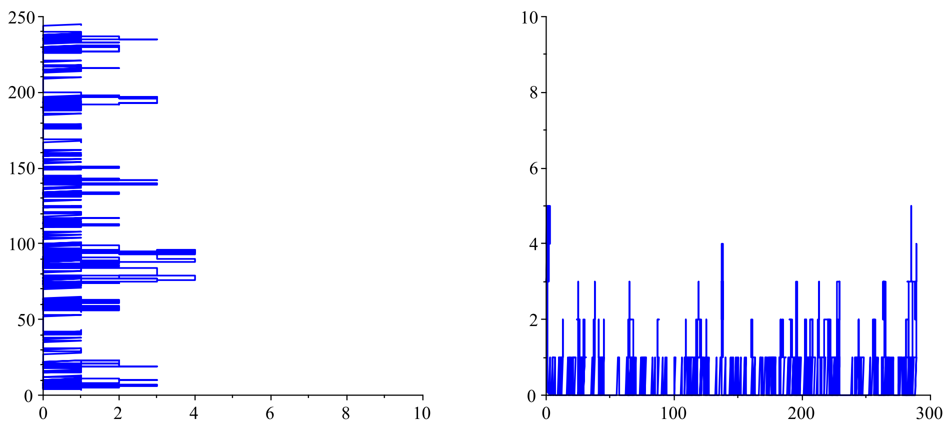

FIGURE 1. Two typical paths, with escape along the vertical and horizontal axes, respectively.

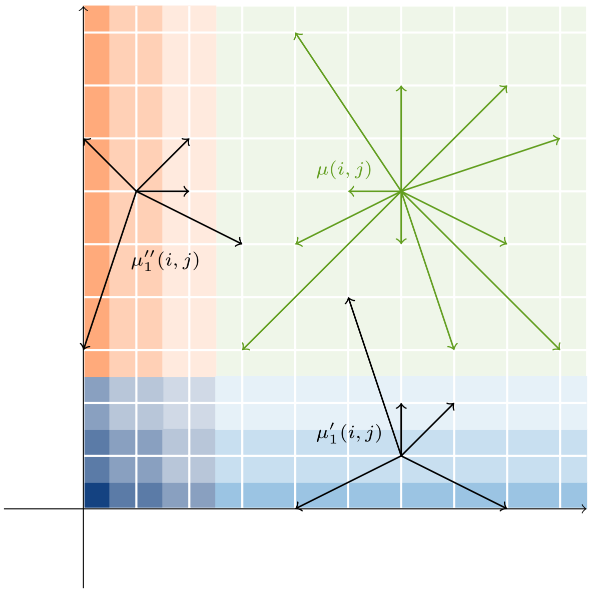

FIGURE 2. Description of the model in the case $k_0 = 3$. Each colored strip is a homogeneity domain for the transition probabilities (1).

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2. THE MODEL

2.1. The main model. Consider a random walk $\{Z(n)\}_{n\geqslant 0} = \{(X(n), Y(n))\}_{n\geqslant 0}$ on $\mathbb{N}^2$ with transition probabilities

$$p\big((i,j) \to (i',j')\big) = \begin{cases} \mu(i'-i, j'-j) & \text{if } i,j \geqslant k_0, \\ \mu'_j(i'-i, j'-j) & \text{if } i \geqslant k_0 \text{ and } 0 \leqslant j < k_0, \\ \mu''_i(i'-i, j'-j) & \text{if } 0 \leqslant i < k_0 \text{ and } j \geqslant k_0, \\ \mu_{ij}(i'-i, j'-j) & \text{if } 0 \leqslant i < k_0 \text{ and } 0 \leqslant j < k_0, \end{cases} \tag{1}$$

where $k_0 > 0$ is a given constant and $\mu, \mu'_j, \mu''_i, \mu_{ij}$ are probability measures on $\mathbb{Z}^2$; see Figure 2. We will assume that the following conditions are satisfied.

Assumption 1. *One has*

(i) $\mu(i,j) = 0$ *if either* $i < -k_0$ *or* $j < -k_0$*;*
(ii) *for any* $j \in \{0, \dots, k_0 - 1\}$*,* $\mu'_j(i', j') = 0$ *if either* $i < -k_0$ *or* $j' < -j$*;*
(iii) *for any* $i \in \{0, \dots, k_0 - 1\}$*,* $\mu''_i(i', j') = 0$ *if either* $j' < -k_0$ *or* $i' < -i$*;*
(iv) *for any* $i, j \in \{0, \dots, k_0 - 1\}$*,* $\mu_{ij}(i', j') = 0$ *if either* $i' < -i$ *or* $j' < -j$*.*

Assumption 2. *There exist positive constants* $\delta, \gamma, C > 0$ *such that*

$$\sup_{(i,j)\in\mathbb{N}^2} \mathbb{E}_{(i,j)}\big(\exp\langle(\delta,\gamma), Z(1) - (i,j)\rangle\big) \leqslant C.$$

2.2. **Auxiliary models.** In addition to our main model $\{Z(n)\}$, we introduce three local random walks, $\{Z_0(n)\}$, $\{Z_1(n)\}$ and $\{Z_2(n)\}$, which correspond to the local behavior of the process $\{Z(n)\}$ far from the boundaries $\mathbb{N}\times\{0,\dots,k_0-1\}$ and $\{0,\dots,k_0-1\}\times\mathbb{N}$; see Figure 3. These secondary processes will be used both in the statements and in the proofs of the main results.

Introduce first the classical random walk $\{Z_0(n)\}$ on $\mathbb{Z}^2$ with homogeneous probabilities of transition

$$p_0\big((i,j)\to(i',j')\big)=\mu(i'-i,j'-j). \tag{2}$$

The mean step (or drift vector) of the random walk $\{Z_0(n)\}$ is $m=(m_1,m_2)$, where

$$m_1=\sum_{(i,j)\in\mathbb{Z}^2} i\mu(i,j) \quad \text{and} \quad m_2=\sum_{(i,j)\in\mathbb{Z}^2} j\mu(i,j). \tag{3}$$

We also define a random walk $Z_1=\{Z_1(n)\}=\{(X_1(n),Y_1(n))\}$ on $\mathbb{N}\times\mathbb{Z}$ with transitions

$$p_1\big((i,j)\to(i',j')\big)=\begin{cases}\mu(i'-i,j'-j) & \text{if } i\geqslant k_0,\\ \mu''_i(i'-i,j'-j) & \text{if } 0\leqslant i<k_0.\end{cases} \tag{4}$$

Similarly, we construct the random walk $Z_2=\{Z_2(n)\}=\{(X_2(n),Y_2(n))\}$ on $\mathbb{Z}\times\mathbb{N}$ with transition probabilities

$$p_2\big((i,j)\to(i',j')\big)=\begin{cases}\mu(i'-i,j'-j) & \text{if } j\geqslant k_0,\\ \mu'_j(i'-i,j'-j) & \text{if } 0\leqslant j<k_0.\end{cases} \tag{5}$$

The local random walk Z_1 (resp. Z_2) describes the behavior of the original walk Z far from the boundary $\{(i,j)\in\mathbb{N}^2 : 0\leqslant j<k_0\}$ (resp. $\{(i,j)\in\mathbb{N}^2 : 0\leqslant i<k_0\}$). We shall assume the following:

Assumption 3. *The random walks Z_0, Z_1, Z_2 and Z are irreducible on their respective state spaces.*

We finally define two one-dimensional Markov chains on $\mathbb{N}$, namely, $\{X_1(n)\}$ and $\{Y_2(n)\}$, with respective transition probabilities

$$p_1(i,i')=\sum_{j'\in\mathbb{Z}} p_1\big((i,0)\to(i',j')\big)=\begin{cases}\sum_{j'\in\mathbb{Z}}\mu(i'-i,j') & \text{if } i\geqslant k_0,\\ \sum_{j'\in\mathbb{Z}}\mu''_i(i'-i,j') & \text{if } 0\leqslant i<k_0,\end{cases}$$

and

$$p_2(j,j')=\sum_{i'\in\mathbb{Z}} p_2\big((0,j)\to(i',j')\big)=\begin{cases}\sum_{i'\in\mathbb{Z}}\mu(i',j'-j) & \text{if } j\geqslant k_0,\\ \sum_{i'\in\mathbb{Z}}\mu'_j(i',j'-j) & \text{if } 0\leqslant j<k_0.\end{cases}$$

The process $\{X_1(n)\}$ (resp. $\{Y_2(n)\}$) is the induced Markov chain relative to the boundary $\{(i,j)\in\mathbb{N}^2 : 0\leqslant i<k_0\}$ (resp. $\{(i,j)\in\mathbb{N}^2 : 0\leqslant j<k_0\}$). We refer to [15] for further definitions and properties of induced Markov chains. The processes X_1 and Y_2 inherit an irreducibility property from Assumption 3. Let us mention the following straightforward result:

Lemma 1. *If $m_1<0$ (resp. $m_2<0$) in (3), the Markov chain $\{X_1(n)\}$ (resp. $\{Y_2(n)\}$) is positive recurrent.*

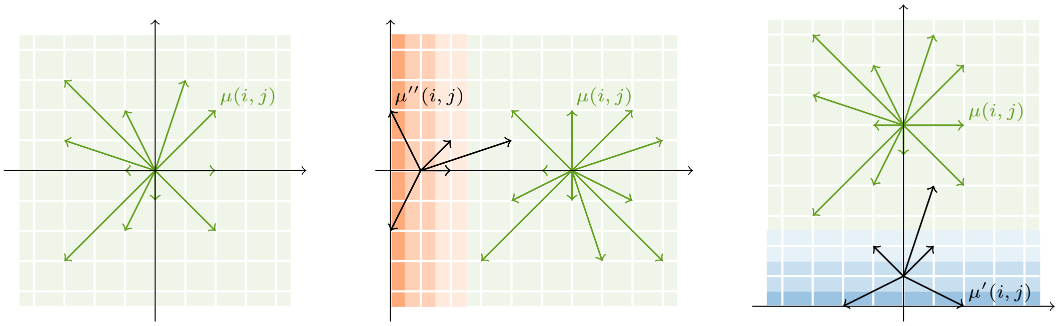

FIGURE 3. Description of the auxiliary models Z_0, Z_1 and Z_2.

Assume that $m_1 < 0$ and $m_2 < 0$, denote by $\{\pi_1(i)\}_{i\in\mathbb{N}}$ and $\{\pi_2(j)\}_{j\in\mathbb{N}}$ the invariant distributions of $\{X_1(n)\}$ and $\{Y_2(n)\}$, and introduce the quantities

$$\left\{\begin{array}{lcl} V_1 & = & \displaystyle\sum_{i=0}^{\infty} \pi_1(i)\mathbb{E}_{(i,0)}\big(Y_1(1)\big), \\ V_2 & = & \displaystyle\sum_{j=0}^{\infty} \pi_2(j)\mathbb{E}_{(0,j)}\big(X_2(1)\big). \end{array}\right. \tag{6}$$

As shown by the following result, the quantity V_1 is well defined whenever $m_1 < 0$. Moreover, V_1 may be interpreted as the velocity of the fluid limit of the local random walk Z_1.

Proposition 2. *If $m_1 < 0$, then the quantity V_1 is well defined. Furthermore, for any $(k,\ell) \in \mathbb{N}\times\mathbb{Z}$, $\mathbb{P}_{(k,\ell)}$-a.s.,*

$$\lim_{n\to\infty} \frac{Y_1(n)}{n} = V_1.$$

A similar, symmetric result holds for V_2. The proof of Proposition 2 is given in Appendix A.

3. Main results

3.1. Statements.

Introduce two further assumptions:

Assumption 4. *The coordinates m_1, m_2 of the drift vector (3) are both negative.*

Assumption 5. *The quantities V_1, V_2 in (6) are positive.*

In particular, Assumption 5 implies that the reflected random walk $\{Z(n)\}$ is transient, with possible escape at infinity along each axis, see Figure 1 and Proposition 3.

We are now ready to state our main results. Recall that the Green function at (i_0,j_0) starting from (i,j) is

$$g\big((i_0,j_0)\to(i,j)\big) = \sum_{n=0}^{\infty} \mathbb{P}_{(i_0,j_0)}\big(Z(n)=(i,j)\big) = \sum_{n=0}^{\infty} p^{(n)}\big((i_0,j_0)\to(i,j)\big). \tag{7}$$

Moreover, $\mathcal{N}_1$ and $\mathcal{N}_2$ will denote the numbers of visits of $\{Z(n)\}$ to the sets $\{0,\ldots,k_0-1\}\times\mathbb{N}$ and $\mathbb{N}\times\{0,\ldots,k_0-1\}$:

$$\mathcal{N}_1=\sum_{n=0}^{\infty}\mathbb{1}_{\{0,\ldots,k_0-1\}\times\mathbb{N}}(Z(n)) \quad\text{and}\quad \mathcal{N}_2=\sum_{n=0}^{\infty}\mathbb{1}_{\mathbb{N}\times\{0,\ldots,k_0-1\}}(Z(n)). \tag{8}$$

The numbers $\mathcal{N}_1$ and $\mathcal{N}_2$ may be infinite. Let finally π_1,π_2 be the invariant distributions of $\{X_1(n)\}$ and $\{Y_2(n)\}$, see Lemma 1 and below.

Theorem 1 (Boundary asymptotics of the Green function). *Assume all Assumptions 1–5. Then, for any $i\in\mathbb{N}$, there exist positive constants C_i and δ_i such that for any $(i_0,j_0)\in\mathbb{N}^2$ and $j\in\mathbb{N}$,*

$$\left|g\big((i_0,j_0)\to(i,j)\big)-\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)\frac{\pi_1(i)}{V_1}\right|\leqslant C_i\exp(-\delta_i(j-j_0)). \tag{9}$$

Similarly, for any $j\in\mathbb{N}$, there exist positive constants C_j' and δ_j' such that for any $(i_0,j_0)\in\mathbb{N}^2$ and $i\in\mathbb{N}$,

$$\left|g\big((i_0,j_0)\to(i,j)\big)-\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)\frac{\pi_2(j)}{V_2}\right|\leqslant C_j'\exp(-\delta_j'(i-i_0)).$$

Theorem 1 shows that asymptotically, the Green function decomposes as a product of two terms, see (9): the first one, $\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)$, is the probability of escape along the horizontal axis (it is a harmonic function); the second part, namely, $\pi_1(i)/V_1$, is the Green function of the half-space random walk Z_1, see Proposition 12.

Theorem 2 (Interior asymptotics of the Green function). *Assume all Assumptions 1–5. As $i+j$ goes to infinity along an angular direction $j/i\to\tan\gamma$, $\gamma\in[0,\frac{\pi}{2}]$, one has*

$$g\big((i_0,j_0)\to(i,j)\big)\sim\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)\frac{\pi_1(i)}{V_1}+\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)\frac{\pi_2(j)}{V_2}. \tag{10}$$

To pursue, we need the following notation: the equation $\sum_{i,j\geqslant -k_0}\mu(i,j)x^iy^j=1$ for $y=1$ and $x>0$ has two solutions, 1 and some $x_1>1$. The same equation with $x=1$ admits two positive solutions, 1 and $y_1>1$ as solutions. See Section 7 for more details. Let now

$$t_0=\tan\gamma_0=\frac{\log x_1}{\log y_1}. \tag{11}$$

Theorem 3 (Martin boundary). *Assume all Assumptions 1–5. The minimal Martin boundary is a union of two points. Moreover, if $t_0\in\mathbb{Q}$ (resp. $t_0\notin\mathbb{Q}$), then the full Martin boundary is homeomorphic to $\mathbb{Z}$ (resp. $\mathbb{R}$).*

3.2. **Remarks on the critical angle γ_0.**

Two examples. Taking first any model which is symmetric in the interior domain, i.e., $\mu(i,j)=\mu(j,i)$ in (1), then one has $x_1=y_1$ and thus $t_0=1$, see (11). In such cases, the full Martin boundary will be discrete, independently of the choice of the boundary parameters. See Figure 4.

On the other hand, take the following simple random walk:

$$\mu(1,0)=\frac{1}{6},\quad \mu(0,-1)=\frac{3}{8},\quad \mu(-1,0)=\frac{1}{3},\quad \mu(0,1)=\frac{1}{8}. \tag{12}$$

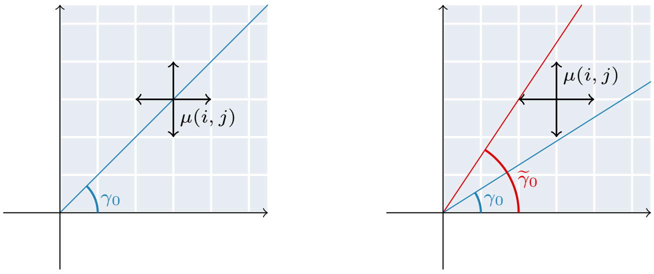

FIGURE 4. Left: for a symmetric model (meaning $\mu(i,j)=\mu(j,i)$), the angle $\gamma_0=\frac{\pi}{4}$ coincides with the angle $\widetilde{\gamma}_0$ made by the drift vector. Right: for the example (12) with (non-symmetric) transition probabilities, these two angles are different ($\gamma_0=\arctan\frac{\log 2}{\log 3}$ and $\widetilde{\gamma}_0=\arctan\frac{3}{2}$).

Then one computes $x_1=2$ and $y_1=3$, and $t_0=\frac{\log 2}{\log 3}$ in (11) is easily shown to be non-rational. The Martin boundary will be continuous.

General remarks on γ_0. Let us first remark that the angle is not the one defined by the drift, which is (see Figure 4)

$$\widetilde{t}_0=\tan\widetilde{\gamma}_0=\frac{x_1}{y_1}.$$

More importantly, the angle γ_0 also appears in two recent articles [24, 17], in the following context. Consider a similar model in the interior domain (the quarter plane), but with a killing on the axes (rather than reflections). The Green functions for the killed random walks are denoted by g_Q (Q for quadrant). Similarly, we introduce the Green functions g_{H_γ} on a half-space H_γ containing the quadrant Q, characterized by its inner normal, which defines an angle $\gamma\in[0,\frac{\pi}{2}]$, see Figure 5. The obvious inclusion $Q\subset H_\gamma$ translates into

$$g_Q\big((i_0,j_0)\to(i,j)\big)\leqslant g_{H_\gamma}\big((i_0,j_0)\to(i,j)\big). \tag{13}$$

Of course, one may also bound the exponential growths:

$$\lim_{i+j\to\infty}\frac{1}{i+j}\log g_Q\big((i_0,j_0)\to(i,j)\big)\leqslant\lim_{i+j\to\infty}\frac{1}{i+j}g_{H_\gamma}\big((i_0,j_0)\to(i,j)\big). \tag{14}$$

The main question addressed in [24, 17] is whether the inequality (14) may be turned into an equality. It turns out that there is, indeed, an angle γ such that (14) becomes an equality, and that this happens precisely for $\gamma=\gamma_0$, with the same γ_0 as in (11).

4. Rough uniform estimates of the Green functions

In this section, we show the following result, proving the uniform boundedness of the Green function (7) along the axes. Let m_1,m_2 be defined in (3) and V_1,V_2 in (6).

Proposition 3. *If $m_1<0$ and $V_1>0$, then the Markov chain $\{Z(n)\}$ is transient and for any $k\in\mathbb{N}$,*

$$\sup_{i,j,\ell\in\mathbb{N}} g\big((i,j)\to(k,\ell)\big)<\infty.$$

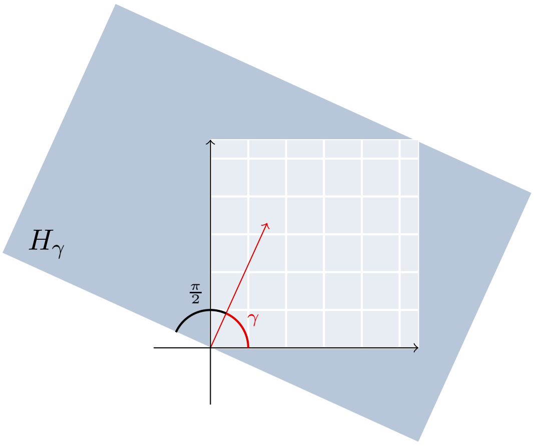

FIGURE 5. Definition of the half-space H_γ, $\gamma \in [0, \frac{\pi}{2}]$.

Similarly, if $m_2 < 0$ *and* $V_2 > 0$, *the Markov chain* $\{Z(n)\}$ *is transient and for any* $\ell \in \mathbb{N}$,

$$\sup_{i,j,k\in\mathbb{N}} g\big((i,j) \to (k,\ell)\big) < \infty.$$

Proof. We give a proof of the first assertion of this lemma. The proof of the second assertion is quite similar. Classically, for any $(i,j),(k,\ell) \in \mathbb{N}^2$,

$$g\big((i,j) \to (k,\ell)\big) = \mathbb{P}_{(i,j)}\big(Z(n) = (k,\ell) \text{ for some } n \geqslant 1\big) g\big((k,\ell) \to (k,\ell)\big) \\ \leqslant g\big((k,\ell) \to (k,\ell)\big),$$

and

$$g\big((k,\ell) \to (k,\ell)\big) = \frac{1}{1 - \mathbb{P}_{(k,\ell)}\big(Z(n) = (k,\ell) \text{ for some } n \geqslant 1\big)}.$$

It is therefore sufficient to show that for any $k \in \mathbb{N}$,

$$\sup_{\ell\in\mathbb{N}} \mathbb{P}_{(k,\ell)}\big(Z(n) = (k,\ell) \text{ for some } n \geqslant 1\big) < 1. \tag{15}$$

To prove the inequality (15), we consider the stopping times

$$\tau(k,\ell) = \inf\{n > 0 : Z(n) = (k,\ell)\}, \tag{16}$$

$$\tau = \inf\{n > 0 : Y(n) < k_0\}, \tag{17}$$

$$\tau_1(k,\ell) = \inf\{n > 0 : Z_1(n) = (k,\ell)\}, \tag{18}$$

$$\tau_1 = \inf\{n > 0 : Y_1(n) < k_0\}, \tag{19}$$

see also Appendix B. Then, for any $(k,\ell) \in \mathbb{N}^2$,

$$\begin{aligned} \mathbb{P}_{(k,\ell)}\big(Z(n) = (k,\ell) \text{ for some } n \geqslant 1\big) &= \mathbb{P}_{(k,\ell)}\big(\tau(k,\ell) < \infty\big) \\ &= \mathbb{P}_{(k,\ell)}\big(\tau < \tau(k,\ell) < \infty\big) + \mathbb{P}_{(k,\ell)}\big(\tau(k,\ell) < \infty \text{ and } \tau(k,\ell) < \tau\big) \\ &\leqslant \mathbb{P}_{(k,\ell)}\big(\tau < \infty\big) + \mathbb{P}_{(k,\ell)}\big(\tau(k,\ell) < \infty \text{ and } \tau(k,\ell) < \tau\big), \end{aligned}$$

where, according to the definition of the local process Z_1,

$$\mathbb{P}_{(k,\ell)}\big(\tau < \infty\big) = \mathbb{P}_{(k,\ell)}\big(\tau_1 < \infty\big)$$

and

$$\mathbb{P}_{(k,\ell)}\big(\tau(k,\ell)<\infty \text{ and } \tau(k,\ell)<\tau\big) = \mathbb{P}_{(k,\ell)}\big(\tau_1(k,\ell)<\infty \text{ and } \tau_1(k,\ell)<\tau\big) \\ \leqslant \mathbb{P}_{(k,\ell)}\big(\tau_1(k,\ell)<\infty\big) = \mathbb{P}_{(k,0)}\big(\tau_1(k,0)<\infty\big).$$

Hence, for any $(k,\ell)\in\mathbb{N}^2$,

$$\mathbb{P}_{(k,\ell)}\big(Z(n)=(k,\ell) \text{ for some } n\geqslant 1\big) \leqslant \mathbb{P}_{(k,\ell)}\big(\tau_1<\infty\big) + \mathbb{P}_{(k,0)}\big(\tau_1(k,0)<\infty\big). \tag{20}$$

Recall now that by Proposition 2, for any $(k,\ell)\in\mathbb{N}^2$, $\mathbb{P}_{(k,\ell)}$-a.s., $\lim_{n\to\infty} Y_1(n)=+\infty$. For any $k\in\mathbb{N}$, the quantity $\inf_{n\in\mathbb{N}} Y_1(n)$ is therefore $\mathbb{P}_{(k,0)}$-a.s. finite, and

$$\mathbb{P}_{(k,0)}\big(\tau_1(k,0)<\infty\big)<1. \tag{21}$$

It follows from the ($\mathbb{P}_{(k,0)}$-a.s.) boundedness of $\inf_{n\in\mathbb{N}} Y_1(n)$ that

$$\lim_{\ell\to\infty} \mathbb{P}_{(k,\ell)}\big(\tau_1<\infty\big) = \lim_{\ell\to\infty} \mathbb{P}_{(k,0)}\big(Y_1(n)<k_0-\ell \text{ for some } n\geqslant 1\big)=0,$$

and using (20), we conclude that for any $k\in\mathbb{N}$,

$$\limsup_{\ell\to\infty} \mathbb{P}_{(k,\ell)}\big(Z(n)=(k,\ell) \text{ for some } n\geqslant 1\big) \leqslant \mathbb{P}_{(k,0)}\big(\tau_1(k,0)<\infty\big).$$

When combined with (21), the last relation proves that for any $k\in\mathbb{N}$ and $N\in\mathbb{N}$ large enough,

$$\sup_{\ell\geqslant N} \mathbb{P}_{(k,\ell)}\big(Z(n)=(k,\ell) \text{ for some } n\geqslant 1\big)<1. \tag{22}$$

The Markov chain $\{Z(n)\}$ being irreducible (see Assumption 3), the last relation proves that $\{Z(n)\}$ is transient, and consequently, for all $\ell\in\mathbb{N}$,

$$\mathbb{P}_{(k,\ell)}\big(Z(n)=(k,\ell) \text{ for some } n\geqslant 1\big)<1. \tag{23}$$

When combined together, (22) and (23) imply (15), and thus Proposition 3. □

5. Preliminary results to the proof of Theorem 1

5.1. Preliminary estimates for hitting probabilities. Consider first the homogeneous random walk $\{Z_0(n)\}=\{(X_0(n),Y_0(n))\}$ on the grid $\mathbb{Z}^2$ with transition probabilities as in (2), see Section 2. We begin our analysis with the following local limit estimate in the large deviation regime.

Lemma 4. *If $m_2<0$, then for any $\theta>0$ small enough, there exists $\delta>0$ such that for any $n>0$, $(i,j)\in\mathbb{Z}^2$ and $\ell\in\mathbb{Z}$,*

$$\mathbb{P}_{(i,j)}\big(Y_0(n)=\ell\big) \leqslant \exp(-\theta(\ell-j)-\delta n). \tag{24}$$

Similarly, if $m_1<0$, then for any $\theta>0$ small enough, there exists $\delta>0$ such that for any $n>0$, $(i,j)\in\mathbb{Z}^2$ and $k\in\mathbb{Z}$,

$$\mathbb{P}_{(i,j)}(X_0(n)=k) \leqslant \exp(-\theta(k-i)-\delta n). \tag{25}$$

Proof. Let us prove the first assertion of this lemma. The proof of the second assertion is totally similar. Consider

$$R(\alpha) = \mathbb{E}_{(0,0)}\big(\exp\langle\alpha, Z_0(1)\rangle\big), \tag{26}$$

the Laplace transform of the increments of the random walk Z_0. Remark that $\nabla R(0) = m = (m_1, m_2)$, and consequently, for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ satisfying the inequality $\langle\alpha, m\rangle < 0$, the function $s \mapsto R(\alpha s)$ is decreasing in a neighborhood of 0. Hence, letting $\alpha_1 = 0$ and $\alpha_2 = \theta > 0$, for $\theta > 0$ small enough, one gets

$$R(\alpha) < 1. \tag{27}$$

Using now Markov inequality, we deduce that

$$\begin{aligned}\mathbb{P}_{(i,j)}\big(Y_0(n) = \ell\big) &\leqslant \mathbb{P}_{(i,j)}\big(\langle Z_0(n), \alpha\rangle \geqslant \theta\ell\big)\\ &\leqslant \exp(-\theta\ell)\mathbb{E}_{(i,j)}\big(\exp\langle\alpha, Z_0(n)\rangle\big)\\ &\leqslant \exp(-\theta\ell)\exp(\theta j)R(\alpha)^n.\end{aligned}$$

The last relation proves (24) with $\delta = -\log R(\alpha) > 0$, and thus Lemma 4. ☐

Consider now the local process Z_1 on $\mathbb{N}\times\mathbb{Z}$ with transition probabilities defined by (4), and the stopping times, for $k \in \mathbb{N}$,

$$T_1(k) = \inf\{n > 0 : X_1(n) = k\}, \tag{28}$$

$$T_1^k = \inf\{n > 0 : X_1(n) \leqslant (k_0 - 1)\vee k\}, \tag{29}$$

where for $j, k \in \mathbb{N}$, we denote $j \vee k = \max\{j, k\}$; see also Appendix B.

Lemma 5. *If $m_1 < 0$, then for any $\theta > 0$, there are $C > 0$ and $\delta > 0$ such that for any $(i,j) \in \mathbb{N}\times\mathbb{Z}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$,*

$$\mathbb{P}_{(i,j)}\big(T_1^k \geqslant n\big) \leqslant C\exp(\theta i - \delta n).$$

Proof. Suppose first that $i \geqslant k_0$. Then for any $j \in \mathbb{Z}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(T_1^k \geqslant n\big) \leqslant \mathbb{P}_{(i,j)}\big(X_0(n) \geqslant 0\big).$$

Hence, by Lemma 4, for any $\theta > 0$, there is $\delta > 0$ such that for any $j \in \mathbb{Z}$, $k \in \mathbb{N}$, $i \geqslant k_0$ and $n \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(T_1^k \geqslant n\big) \leqslant \sum_{\ell=0}^{\infty}\mathbb{P}_{(i,j)}\big(X_0(n) = \ell\big) \leqslant \frac{\exp(\theta i - 2\delta n)}{1 - \exp\theta}.$$

For $i \geqslant k_0$, Lemma 5 is therefore proved.

To prove this lemma in the pending cases $0 \leqslant i < k_0$, we notice that in this case,

$$\begin{aligned}\mathbb{P}_{(i,j)}\big(T_1^k \geqslant n\big) &= \sum_{i' \geqslant k_0, j' \in \mathbb{Z}}\mathbb{P}_{(i,j)}\big(Z_1(1) = (i', j')\big)\mathbb{P}_{(i',j')}\big(T_1^k \geqslant n - 1\big)\\ &\leqslant \sum_{i' \geqslant k_0, j' \in \mathbb{Z}}\mathbb{P}_{(i,j)}\big(Z_1(1) = (i', j')\big)\mathbb{P}_{(i',j')}\big(X_0(n-1) \geqslant 0\big),\end{aligned}$$

and hence, using again Lemma 4, we conclude that for any $\theta > 0$ small enough, there is $\delta > 0$ such that for all $(i, j) \in \mathbb{N} \times \mathbb{Z}$ with $i < k_0$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}_{(i,j)}\big(T_1^k \geqslant n\big) &\leqslant \widetilde{C} \sum_{i' \geqslant k_0, j' \in \mathbb{Z}} \mathbb{P}_{(i,j)}\big(Z_1(1) = (i', j')\big) \exp(\theta i' - \delta n) \\ &= \widetilde{C} \exp(\theta i - \delta n) \mathbb{E}_{(i,j)}\big(\exp(\theta(X_1(1) - i))\big),\end{aligned}$$

with $\widetilde{C} = e^{\theta+\delta}/(1 - e^{\theta})$. To complete the proof of Lemma 4, it is therefore sufficient to choose $\theta > 0$ small enough, so that

$$\sup_{(i,j) \in \mathbb{N}^2} \mathbb{E}_{(i,j)}\big(\exp(\theta(X_1(1) - i)\big) < \infty.$$

Such a small $\theta > 0$ exists thanks to Assumption 2. □

As a straightforward consequence of Lemma 5, one deduces the following result:

Corollary 6. *If $m_1 < 0$, then for any $\theta > 0$, there are $C > 0$ and $\delta > 0$ such that for any $(i, j) \in \mathbb{N} \times \mathbb{Z}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_{(i,j)}\big(\exp(\delta T_1^k)\big) \leqslant C \exp(\theta i).$$

Proof. Indeed, by Lemma 5, for any $\theta > 0$, there are $C >$ and $\delta > 0$ such that for any $(i, j) \in \mathbb{N} \times \mathbb{Z}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(T_1^k \geqslant n\big) \leqslant C \exp(\theta i - 2\delta n),$$

and consequently,

$$\mathbb{E}_{(i,j)}\big(\exp(\delta T_1^k)\big) \leqslant C \exp(\theta i) \sum_{n=1}^{\infty} e^{\delta n} \mathbb{P}_{(i,j)}\big(T_1^k \geqslant n\big) \leqslant C \frac{\exp(\theta i)}{1 - \exp \delta}.$$ □

Lemma 7. *If $m_1 < 0$, then for any $\theta > 0$ and $k \in \mathbb{N}$, there are $C > 0$ and $\delta > 0$ such that for any $(i, j) \in \mathbb{N} \times \mathbb{Z}$ and $n \in \mathbb{N}$,*

$$\mathbb{P}_{(i,j)}\big(T_1(k) \geqslant n\big) \leqslant C \exp(\theta i - \delta n). \tag{30}$$

Proof. Consider the sequence of stopping times $\{t_n\}_{n \geqslant 0}$ defined by

$$t_0 = 0, \quad t_1 = T_1^k \quad \text{and} \quad t_{n+1} = \inf\{s > t_n : X_1(s) < (k_0 - 1) \vee k\}.$$

They are finite almost surely, as $m_1 < 0$. By the strong Markov property, the sequence $\{X_1(t_n)\}_{n \geqslant 1}$ is a Markov chain on the finite state space $\{0, \ldots, (k_0 - 1) \vee k\}$. It is irreducible because the Markov chain X_1 is irreducible, and according to the definition of the stopping times T_1^k and $T_1(k)$,

$$T_1(k) = t_\nu, \quad \text{with} \quad \nu = \inf\{n > 0 : X(t_n) = k\}.$$

Hence, for any $(i, j) \in \mathbb{N} \times \mathbb{Z}$, $n \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(T_1(k) \geqslant n\big) = \mathbb{P}_{(i,j)}\big(t_\nu \geqslant n\big) \leqslant \mathbb{P}_{(i,j)}\big(\nu > s\big) + \mathbb{P}_{(i,j)}\big(t_\nu \geqslant n, \nu \leqslant s\big),$$

and by Perron-Frobenius theorem, there exist $\widetilde{C} > 0$ and $\widetilde{\delta} > 0$ such that for any $n \in \mathbb{N}$ and $(i, j) \in \mathbb{N} \times \mathbb{Z}$,

$$\mathbb{P}_{(i,j)}\big(\nu > s\big) \leqslant \widetilde{C} \exp(-\widetilde{\delta} s).$$

Moreover, the sequence $\{t_n\}$ being increasing, one has

$$\mathbb{P}_{(i,j)}\big(t_\nu \geqslant n, \nu \leqslant s\big) \leqslant \mathbb{P}_{(i,j)}\big(t_s \geqslant n, \nu \leqslant s\big) \leqslant \mathbb{P}_{(i,j)}\big(t_s \geqslant n\big),$$

and consequently, for any $(i,j) \in \mathbb{N} \times \mathbb{Z}$, $n \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(T_1(k) \geqslant n\big) \leqslant \widetilde{C} \exp(-\widetilde{\delta} s) + \mathbb{P}_{(i,j)}\big(t_s \geqslant n\big). \tag{31}$$

Remark finally that by Markov inequality, for any $\delta > 0$, $i \in \{0, \ldots, (k_0 - 1) \vee k\}$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(t_s \geqslant n\big) \leqslant \exp(-\delta n) \mathbb{E}_{(i,j)}\big(\exp(\delta t_s)\big),$$

where according to the definition of the sequence $\{t_n\}$, by the strong Markov property,

$$\mathbb{E}_{(i,j)}\big(\exp(\delta t_s)\big) \leqslant \mathbb{E}_{(i,j)}\big(\exp(\delta T_1^k)\big) \left(\sup_{\substack{0 \leqslant i' \leqslant (k_0-1) \vee k \\ j' \in \mathbb{Z}}} \mathbb{E}_{(i',j')}\big(\exp(\delta T_1^k)\big) \right)^{s-1}.$$

Applying now Corollary 6, one deduces that for any $\theta > 0$, there exist $\widehat{C} > 0$ and $\widehat{\delta} > 0$ such that for any $(i,j) \in \mathbb{N} \times \mathbb{Z}$, $n \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(t_s \geqslant n\big) \leqslant \widehat{C}^s \exp(\theta i - \widehat{\delta} n). \tag{32}$$

When combined together, the inequalities (31) and (32) prove that for any $\theta > 0$, there are $\widetilde{C}, \widehat{C} > 0$ and $\widetilde{\delta}, \widehat{\delta} > 0$ such that for any $(i,j) \in \mathbb{N} \times \mathbb{Z}$, $n \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\mathbb{P}_{(i,j)}\big(T_1(k) \geqslant n\big) \leqslant \widetilde{C} \exp(-\widetilde{\delta} s) + \widehat{C}^s \exp(\theta i - \widehat{\delta} n)$$

and hence, choosing $\frac{\widehat{\delta}}{2} \log(\widehat{C}) \leqslant s \leqslant \frac{\widehat{\delta}}{2} \log(\widehat{C}) + 1$, one obtains (30), with some $C > 0$ and $\delta = \min\{\widehat{\delta}, \widetilde{\delta}\widehat{\delta} / \log(\widehat{C})\}/2$. □

Lemma 8. *If $m_1 < 0$, then for any $\theta > 0$ and $k \in \mathbb{N}$, there exist $C > 0$ and $\delta > 0$ such that for any $i \in \mathbb{N}$ and $n \in \mathbb{N}$,*

$$\mathbb{P}_{(i,0)}\big(|Y_1(T_1(k))| \geqslant n\big) \leqslant C \exp(\theta i - \delta n). \tag{33}$$

Proof. For any $i \in \mathbb{N}$, $n \in \mathbb{N}$ and $\varkappa > 0$,

$$\begin{aligned} \mathbb{P}_{(i,0)}\big(|Y_1(T_1(k))| \geqslant n\big) &\leqslant \mathbb{P}_{(i,0)}\big(|Y_1(T_1(k))| \geqslant n, T_1(k) \leqslant \varkappa n\big) + \mathbb{P}_{(i,0)}\big(T_1(k) > \varkappa n\big) \\ &\leqslant \sum_{1 \leqslant s \leqslant \varkappa n} \mathbb{P}_{(i,0)}\big(|Y_1(s)| \geqslant n\big) + \mathbb{P}_{(i,0)}\big(T_1(k) > \varkappa n\big). \end{aligned}$$

Hence, for any $\widetilde{\delta} > 0$, using Markov inequality, one gets

$$\mathbb{P}_{(i,0)}\big(|Y_1(T_1(k))| \geqslant n\big) \leqslant \exp(-\widetilde{\delta} n) \sum_{1 \leqslant s \leqslant \varkappa n} \mathbb{E}_{(i,0)}\big(\exp(\widetilde{\delta} |Y_1(s)|)\big) + \mathbb{P}_{(i,0)}\big(T_1(k) > \varkappa n\big).$$

On the one hand, using Assumption 2, for $\widetilde{\delta} > 0$ small enough,

$$\mathbb{E}_{(i,0)}\big(\exp(\widetilde{\delta} |Y_1(s)|)\big) \leqslant \widetilde{C}^s,$$

with some $\widetilde{C} > 0$ not depending on $i \in \mathbb{N}$ and $s \in \mathbb{N}$. On the other hand, by Lemma 7, for any $\theta > 0$ and $k \in \mathbb{N}$, there exist $\widehat{C} > 0$ and $\widehat{\delta} > 0$ such that for any $i \in \mathbb{N}$, $n \in \mathbb{N}$ and $\varkappa > 0$,

$$\mathbb{P}_{(i,0)}\big(T_1(k) > \varkappa n\big) \leqslant \widehat{C} \exp(\theta i - \widehat{\delta} \varkappa n).$$

When combined together, these inequalities imply that

$$\mathbb{P}_{(i,0)}\big(|Y_1(T_1(k))| \geqslant n\big) \leqslant \exp(-\widetilde{\delta} n)\frac{\widetilde{C}^{\varkappa n}}{\widetilde{C}-1} + \widehat{C}\exp(\theta i - \widehat{\delta}\varkappa n),$$

and hence, letting $\varkappa = \widetilde{\delta}\log(\widetilde{C})/2$, one gets (33) with $\delta = \min\{\widetilde{\delta}, \widehat{\delta}\varkappa\}/2$ and some $C > 0$. □

Lemma 9. *If $m_2 < 0$, then for any $k \geqslant 0$, there exist $C > 0$, $\theta > 0$ and $\widetilde{\theta} > 0$ such that for any $i \geqslant k_0$ and $j, \ell \in \mathbb{Z}$,*

$$\mathbb{P}_{(i,j)}\big(Y_1(T_1^k) = \ell\big) \leqslant C\exp(-\theta(\ell - j) - \widetilde{\theta} i). \tag{34}$$

Proof. According to the definitions of the local process Z_1 and the stopping time T_1^k in (29), it follows from Lemma 4 that there exist $\theta > 0$ and $\delta > 0$ such that for any $n > 0$, $k \geqslant 0$, $i \geqslant k_0$ and $j, \ell \in \mathbb{Z}$,

$$\mathbb{P}_{(i,j)}\big(Y_1(T_1^k) = \ell, T_1^k = n\big) \leqslant \mathbb{P}_{(i,j)}\big(Y_0(n) = \ell\big) \leqslant \exp(-\theta(\ell - j) - \delta n).$$

Moreover, by Assumption 1, the jumps of the random walk Z_0 are bounded from below, i.e., whenever $i' - i < -k_0$ or $j' - j < -k_0$,

$$\mathbb{P}_{(i,j)}\big(Z_0(1) = (i', j')\big) = \mu(i' - i, j' - j) = 0.$$

Hence, $\mathbb{P}_{(i,j)}\big(T_1^k = n\big) = 0$ for all $i > k_0 n + k_0 \vee k$, and consequently,

$$\begin{aligned}
\mathbb{P}_{(i,j)}\big(Y_1(T_1^k) = \ell\big) &= \sum_{n \geqslant (i - k_0 \vee k)/k_0} \mathbb{P}_{(i,j)}\big(Y_1(T_1^k) = \ell, \tau_1 = n\big) \\
&\leqslant \sum_{n \geqslant (i - k_0 \vee k)/k_0} \exp\big(-\theta(\ell - j) - \delta n\big) \\
&\leqslant (1 - e^{-\delta})^{-1}\exp\big(-\theta(\ell - j) - \delta(i - k_0 \vee k)/k_0\big).
\end{aligned}$$

The last relation proves (34) with $\widetilde{\theta} = \delta/k_0$ and $C = (1 - e^{-\delta})^{-1}\exp(\delta(k_0 \vee k))$. □

5.2. **Preliminary results for local random walks.** In order to prove Theorem 1, we first investigate the local process Z_1 and its Green function

$$g_1\big((i,j) \to (i', j')\big) = \sum_{n=0}^{\infty} \mathbb{P}_{(i,j)}\big(Z_1(n) = (i', j')\big).$$

The corresponding generating functions are

$$\mathcal{G}^1_{(i,j)\to(k,\cdot)}(y) = \sum_{\ell=-\infty}^{\infty} g_1\big((i,j) \to (k,\ell)\big)y^{\ell}, \tag{35}$$

for any $k \in \mathbb{N}$. We define the Laurent series

$$\Phi^1_{(i,j)\to(k,\cdot)}(y) = \sum_{\ell=-\infty}^{\infty} \mathbb{P}_{(i,j)}\big(Y_1(T_1(k)) = \ell\big)y^{\ell}, \tag{36}$$

where the stopping time $T_1(k)$ is defined by (28).

Proposition 10. *Assume that $m_1 < 0$ and $V_1 > 0$. Then for all $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ such that for any $(i,j) \in \mathbb{N}^2$, the function $\mathcal{G}^1_{(i,j)\to(k,\cdot)}(y)$ in (35) is analytic in the annulus $\{1-\varepsilon_k < |y| < 1\}$ and can be meromorphically continued to $\{1-\varepsilon_k < |y| < 1+\varepsilon_k\}$, where it satisfies*

$$\mathcal{G}^1_{(i,j)\to(k,\cdot)}(y) = \frac{\Phi^1_{(i,j)\to(k,\cdot)}(y)}{1-\Phi^1_{(k,0)\to(k,\cdot)}(y)}, \tag{37}$$

see (36), and admits a unique simple pole at the point $y = 1$, with residue $\pi_1(k)/V_1$.

Proof. By Lemma 8, for some $\widehat{\varepsilon}_k > 0$, the series (36) converges normally in the annulus $\{1-\widehat{\varepsilon}_k < |y| < 1+\widehat{\varepsilon}_k\}$, with

$$\Phi^1_{(k,j)\to(k,\cdot)}(1) = 1.$$

Moreover, using Lemma 24 in Appendix A,

$$\frac{d}{dy}\Phi^1_{(k,0)\to(k,\cdot)}(1) \;=\; \mathbb{E}_{(k,0)}\big(Y_1(\tau_1(k))\big) = V_1/\pi_1(k) \;>\; 0. \tag{38}$$

Recall furthermore that the local process Z_1 is irreducible in its state space $\mathbb{N}\times\mathbb{Z}$, and hence for any $k \in \mathbb{N}$, the positive matrix $\big(\mathbb{P}_{(k,j)}\big(Y_1(\tau_1(k)) = \ell\big)\big)_{j,\ell\in\mathbb{Z}}$ is also irreducible. This proves that the sequence of integers $\ell \in \mathbb{Z}$ for which $\mathbb{P}_{(k,j)}\big(Y_1(\tau_1(k)) = \ell\big) > 0$ is aperiodic and consequently, for any point y in the unit circle $|y| = 1$ such that $y \neq 1$,

$$\left|\Phi^1_{(k,j)\to(k,\cdot)}(1)\right| < 1.$$

This proves that for some $\widehat{\varepsilon}_k > 0$, the right-hand side of (37) is meromorphic in the annulus $\{1-\widehat{\varepsilon}_k < |y| < 1+\widehat{\varepsilon}_k\}$ and has there a unique, simple pole at the point $y = 1$, with residue $\pi_1(k)/V_1$.

Now, to complete the proof of our proposition, it is sufficient to show that for some $\widetilde{\varepsilon}_k > 0$ small enough and any y in the annulus $\{1-\widetilde{\varepsilon}_k < |y| < 1\}$, (37) holds. To that aim, we remark that thanks to (38), for $\widetilde{\varepsilon}_k > 0$ small enough,

$$\Phi^1_{(k,0)\to(k,\cdot)}(y) < 1 \quad \text{whenever } y \in \mathbb{R} \text{ and } 1-\widetilde{\varepsilon}_k < y < 1,$$

see for instance [2, Eq. (35)]. This proves that the series

$$\sum_{s=-\infty}^{\infty} \mathbb{P}_{(i,j)}\big(Y_1(\tau_1(k)) = \ell\big)y^s \sum_{n=0}^{\infty}\left(\sum_{\ell=-\infty}^{\infty} \mathbb{P}_{(k,0)}\big(Y_1(\tau_1(k)) = \ell\big)y^\ell\right)^n$$

converges normally in $\{1-\widetilde{\varepsilon}_k < |y| < 1\}$, and hence, by Fubini's theorem, for a sequence of stopping times defined by

$$t_0 = 0, \quad t_1 = \tau_1(k) \quad \text{and} \quad t_{n+1} = \inf\{s > t_n : X_1(s) = k\},$$

we have

$$\begin{aligned}\frac{\Phi^1_{(i,j)\to(k,\cdot)}(y)}{1-\Phi^1_{(k,0)\to(k,\cdot)}(y)} &= \sum_{s=-\infty}^{\infty} \mathbb{P}_{(i,j)}\big(Y_1(\tau_1(k)) = s\big)y^s \sum_{n=0}^{\infty}\left(\sum_{\ell=-\infty}^{\infty} \mathbb{P}_{(k,0)}\big(Y_1(\tau_1(k)) = \ell\big)y^\ell\right)^n \\ &= \sum_{\ell=-\infty}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}_{(i,j)}\big(Y_1(t_{n+1}) = \ell\big)y^\ell,\end{aligned}$$

whenever $\{1-\widetilde{\varepsilon}_k < |y| < 1\}$. Since for any $(i,j)\in\mathbb{N}\times\mathbb{Z}$ and $\ell\in\mathbb{Z}$,

$$\sum_{n=0}^{\infty}\mathbb{P}_{(i,j)}\big(Y_1(t_{n+1})=\ell\big)=g_1\big((i,j)\to(k,\ell)\big),$$

the last relation implies (37). □

As a straightforward consequence of Proposition 10, one obtains the following:

Corollary 11. *Assume that $m_1<0$ and $V_1>0$. Then for any $k\in\mathbb{N}$, there exists $\delta_k>0$ such that for any $i,j,\ell\in\mathbb{N}$,*

$$\left|g_1\big((i,j)\to(k,\ell)\big)-\frac{\pi_1(k)}{V_1}\right|\leqslant C_{i,k}\exp\big(-\delta_k(\ell-j)\big),$$

with some constant $C_{i,k}>0$ independent of $j,\ell\in\mathbb{N}$.

We will need moreover the following refinement of this result.

Proposition 12. *Assume that $m_1<0$, $m_2<0$ and $V_1>0$. Then for any $k\in\mathbb{N}$, there exist $\widetilde{\delta}_k>0$ and $C_k>0$ such that for any $i,j,\ell\in\mathbb{N}$,*

$$\left|g_1\big((i,j)\to(k,\ell)\big)-\frac{\pi_1(k)}{V_1}\right|\leqslant C_k\exp\big(-\widetilde{\delta}_k(\ell-j)\big). \tag{39}$$

Proof. Consider the stopping time T_1^k as defined in (29) and let $(i,j),(k,\ell)\in\mathbb{N}^2$ with $i>k_0$. Then

$$\begin{aligned}g_1\big((i,j)\to(k,\ell)\big)&=g_1\big((i,0)\to(k,\ell-j)\big)\\&=\sum_{(i',j')\in\{0,\dots,(k_0-1)\vee k\}\times\mathbb{Z}}\mathbb{P}_{(i,0)}\big(Z_1(T_1^k)=(i',j')\big)g_1\big((i',j')\to(k,\ell-j)\big).\end{aligned}$$

Moreover, since $m_1<0$, we have

$$\sum_{(i',j')\in\{0,\dots,(k_0-1)\vee k\}\times\mathbb{Z}}\mathbb{P}_{(i,0)}\big(Z_1(T_1^k)=(i',j')\big)=\mathbb{P}_{(i,0)}\big(T_1^k<\infty\big)=1,$$

and consequently

$$\begin{aligned}&\left|g_1\big((i,j)\to(k,\ell)\big)-\frac{\pi_1(k)}{V_1}\right|\\&\quad\leqslant\sum_{(i',j')\in\{0,\dots,(k_0-1)\vee k\}\times\mathbb{Z}}\mathbb{P}_{(i,0)}\big(Z_1(T_1^k)=(i',j')\big)\left|g_1\big((i',j')\to(k,\ell-j)\big)-\frac{\pi_1(k)}{V_1}\right|.\end{aligned}$$

In order to derive (39), we now split the right-hand side of the above inequality into two parts:

$$\Sigma_1=\sum_{i'\in\{0,\dots,(k_0-1)\vee k\},j'\geqslant(\ell-j)/2}\mathbb{P}_{(i,0)}\big(Z_1(T_1^k)=(i',j')\big)\left|g_1\big((i',j')\to(k,\ell-j)\big)-\frac{\pi_1(k)}{V_1}\right|,$$

$$\Sigma_2=\sum_{i'\in\{0,\dots,(k_0-1)\vee k\},j'<(\ell-j)/2}\mathbb{P}_{(i,0)}\big(Z_1(T_1^k)=(i',j')\big)\left|g_1\big((i',j')\to(k,\ell-j)\big)-\frac{\pi_1(k)}{V_1}\right|.$$

To estimate Σ_1, we combine the straightforward inequality

$$\left|g_1\big((i',j')\to(k,\ell-j)\big)-\frac{\pi_1(k)}{V_1}\right|\leqslant g_1\big((k,\ell-j)\to(k,\ell-j)\big)+\frac{1}{V_1}=g_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}$$

with Lemma 9:

$$\begin{aligned}\Sigma_1&\leqslant\sum_{i'\in\{0,\ldots,(k_0-1)\vee k\},j'\geqslant(\ell-j)/2}\mathbb{P}_{(i,0)}\big(Z_1(T_1^k)=(i',j')\big)\left(g_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right)\\&\leqslant\sum_{j'\geqslant(\ell-j)/2}\mathbb{P}_{(i,0)}\big(Y_1(\tau_1)=j'\big)\left(g_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right)\\&\leqslant C\left(g_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right)\sum_{j'\geqslant(\ell-j)/2}\exp\big(-\theta j'-\widetilde{\theta}i\big)\\&\leqslant C\left(g_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right)(1-e^{-\theta})^{-1}\exp\big(-\theta(\ell-j)/2\big).\end{aligned}$$

In order to estimate Σ_2, we use Corollary 11:

$$\begin{aligned}\Sigma_2&\leqslant\sum_{i'\in\{0,\ldots,(k_0-1)\vee k\},j'<(\ell-j)/2}\mathbb{P}_{(i,j)}\big(Z_1(T_1^k)=(i',j')\big)C_{i',k}\exp\big(-\delta_k(\ell-j-j')\big)\\&\leqslant\exp\big(-\delta_k(\ell-j)/2\big)\max_{i'\in\{0,\ldots,(k_0-1)\vee k\}}C_{i',k}.\end{aligned}$$

When combined together, these estimates prove (39) with $\widetilde{\delta}_k=\min\{\theta,\delta_k\}$ and

$$C_k=C\left(g_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right)(1-e^{-\theta})^{-1}+\max_{i'\in\{0,\ldots,(k_0-1)\vee k\}}C_{i',k}.\qquad\square$$

5.3. **Preliminary results for killed local random walks.** In this part, we investigate the Green functions of killed local random walks. More specifically, consider the stopping time

$$\tau_1=\inf\{n>0:Y_1(n)<k_0\}$$

as defined in (19), and denote by $\widehat{Z}_1$ a copy of the local random walk Z_1 killed at time τ_1. The Green function of $\widehat{Z}_1$ is defined by

$$\widehat{g}_1\big((i,j)\to(i',j')\big)=\sum_{n=0}^{\infty}\mathbb{P}_{(i,j)}\big(Z_1(n)=(i',j'),\tau_1>n\big).\tag{40}$$

According to this definition, $\widehat{g}_1\big((i,j)\to(i',j')\big)=0$ if either $j<k_0$ or $j'<k_0$. The main result of this section is the following:

Proposition 13. *Assume $m_1<0$, $m_2<0$ and $V_1>0$. Then for any $k\in\mathbb{N}$, there exist $\delta_k>0$ and $C_k'>0$ such that for any $i\in\mathbb{N}$ and $j,\ell\geqslant k_0$,*

$$\left|\widehat{g}_1\big((i,j)\to(k,\ell)\big)-\frac{\pi_1(k)}{V_1}\mathbb{P}_{(i,j)}\big(\tau_1=\infty\big)\right|\leqslant C_k'\exp\big(-\delta_k(\ell-j)\big).\tag{41}$$

Proof. To prove (41), we combine Proposition 12 with the following two identities:

$$g_1\big((i,j)\to(k,\ell)\big) = \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} \mathbb{P}_{(i,j)}\big(Z_1(\widehat{\tau}_1)=(i',j')\big) g_1\big((i',j')\to(k,\ell)\big) + \widehat{g}_1\big((i,j)\to(k,\ell)\big),$$

$$\mathbb{P}_{(i,j)}\big(\tau_1<\infty\big) = \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} \mathbb{P}_{(i,j)}\big(Z_1(\widehat{\tau}_1)=(i',j')\big).$$

Using these relations, one gets

$$\begin{aligned}\left|\widehat{g}_1\big((i,j)\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i,j)}\big(\tau_1=\infty\big)\right| &\leqslant \left|g_1\big((i,j)\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\right| \\ &\quad + \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} \mathbb{P}_{(i_0,j_0)}\left(Z_1(\widehat{\tau}_1)=(i',j')\right)\left|g_1\big((i',j')\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\right|.\end{aligned}$$

Consequently, by Proposition 12, for any $k\in\mathbb{N}$, there exist $\delta_k>0$ and $C_k>0$ such that for any $i,j,\ell\in\mathbb{N}$,

$$\begin{aligned}&\left|\widehat{g}_1\big((i,j)\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i,j)}\big(\tau_1=\infty\big)\right| \\ &\qquad\leqslant C_k\exp\big(-\delta_k(\ell-j)\big) + \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} \mathbb{P}_{(i,j)}\left(Z_1(\widehat{\tau}_1)=(i',j')\right) C_k\exp\big(-\delta_k(\ell-j')\big) \\ &\qquad\leqslant C_k\exp\big(-\delta_k(\ell-j)\big)\big(1+\exp(\delta_k(k_0-1))\big).\end{aligned}$$

Proposition 13 is therefore proved. ☐

5.4. **Preliminary results for the original random walk.** In this part, we investigate the Green function $g\big((i,j)\to(k,\ell)\big)$ of the original random walk Z, see Section 2. Consider the stopping time

$$\tau=\inf\{n>0 : Y(n)<k_0\}$$

as defined in (17), see also Appendix B, and let

$$\widehat{g}_1^+\big((i,j)\to(k,\ell)\big) = \sum_{n=0}^{\infty}\mathbb{P}_{(i,j)}\big(Z(n)=(k,\ell),\tau>n\big).$$

It represents the Green function of the process Z before approaching the boundary axes $\{(i,j)\in\mathbb{N}^2 : j<k_0\}$. According to the definition of the killed random walk $\widehat{Z}_1$ and the corresponding Green function $\widehat{g}_1$ in (40), one has for any $(i,j)\in\mathbb{N}^2$, $k\in\mathbb{N}$ and $\ell\geqslant k_0$,

(42)
$$\widehat{g}_1^+\big((i,j)\to(k,\ell)\big) = \begin{cases}\widehat{g}_1\big((i,j)\to(k,\ell)\big) & \text{if } j\geqslant k_0,\\ \sum_{i'\geqslant 0, j'\geqslant k_0}\mathbb{P}_{(i,j)}\big(Z(1)=(i',j')\big)\widehat{g}_1\big((i',j')\to(k,\ell)\big) & \text{if } j<k_0.\end{cases}$$

As a consequence of Proposition 13, we obtain:

Proposition 14. *If $m_1<0$, $m_2<0$ and $V_1>0$, then for any $k\in\mathbb{N}$, there exist $\delta_k''>0$ and $C_k''>0$ such that for any $i,j\in\mathbb{N}$ and $\ell\geqslant k_0$,*

$$\left|\widehat{g}_1^+\big((i,j)\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i,j)}\big(\tau=\infty\big)\right| \leqslant C_k''\exp\big(-\delta_k''(\ell-j)\big). \tag{43}$$

Proof. We first assume that $j,\ell \geqslant k_0$. In this case, one has $\widehat{g}_1^+ = \widehat{g}_1$ for any $i,k \in \mathbb{N}$, by (42), and consequently (43) is already proved by Proposition 13.

Suppose now that $i,k \in \mathbb{N}$, $\ell \geqslant k_0$ and $0 \leqslant j < k_0$. Then by (42),

$$\widehat{g}_1^+\big((i,j)\to(k,\ell)\big) = \sum_{i'\geqslant 0, j'\geqslant k_0} \mathbb{P}_{(i,j)}\big(Z(1)=(i',j')\big)\widehat{g}_1\big((i',j')\to(k,\ell)\big).$$

Since we have

$$\mathbb{P}_{(i,j)}(\tau=\infty) = \sum_{i'\geqslant 0, j'\geqslant k_0} \mathbb{P}_{(i,j)}\big(Z(1)=(i',j')\big)\mathbb{P}_{(i',j')}(\tau=\infty),$$

it follows that

$$\begin{aligned}&\left|\widehat{g}_1^+\big((i,j)\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i,j)}(\tau=\infty)\right| \\ &\qquad \leqslant \sum_{i'\geqslant 0, j'\geqslant k_0} \mathbb{P}_{(i,j)}\big(Z(1)=(i',j')\big)\left|\widehat{g}_1\big((i',j')\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i',j')}(\tau=\infty)\right|.\end{aligned}$$

In order to derive (43), we now split the right-hand side above into two parts:

$$\Sigma_1 = \sum_{i'\geqslant 0, j'>\ell/2} \mathbb{P}_{(i,j)}\big(Z(1)=(i',j')\big)\left|\widehat{g}_1\big((i',j')\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i',j')}(\tau=\infty)\right|,$$

$$\Sigma_2 = \sum_{i'\geqslant 0, k_0\leqslant j'\leqslant\ell/2} \mathbb{P}_{(i,j)}\big(Z(1)=(i',j')\big)\left|\widehat{g}_1\big((i',j')\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i',j')}(\tau=\infty)\right|.$$

Using first the inequality (see the proof of Proposition 3 for similar computations)

$$\widehat{g}_1\big((i',j')\to(k,\ell)\big) \leqslant \widehat{g}_1\big((k,\ell)\to(k,\ell)\big) = \widehat{g}_1\big((k,0)\to(k,0)\big),$$

we deduce that

$$\Sigma_1 \leqslant \mathbb{P}_{(i,j)}\big(Y(1)>\ell/2\big)\left(\widehat{g}_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right).$$

Moreover, applying Markov inequality and using Assumption 2, we obtain for $\alpha=(0,\delta)$ with $\delta>0$ small enough

$$\mathbb{P}_{(i,j)}\big(Y(1)>\ell/2\big) \leqslant \exp(-\delta\ell/2)\mathbb{E}_{(i,j)}\big(\exp\langle\alpha,Z(1)\rangle\big) \leqslant C_0\exp\big(-\delta\ell/2+\delta(k_0-1)\big).$$

Therefore,

$$\Sigma_1 \leqslant C_0\left(\widehat{g}_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right)\exp\big(-\delta\ell/2+\delta(k_0-1)\big). \tag{44}$$

Using now Proposition 13, we get

$$\Sigma_2 \leqslant \sum_{i'\geqslant 0, k_0\leqslant j'\leqslant\ell/2} \mathbb{P}_{(i,j)}\big(Z(1)=(i',j')\big)C_k\exp\big(-\delta_k(\ell-j')\big) \leqslant C_k\exp\big(-\delta_k\ell/2\big), \tag{45}$$

When combined together, (44) and (45) prove (43) with $\delta_k''=\min\{\delta_k,\delta\}/2$ and

$$C_k'' = C_k + C_0\left(\widehat{g}_1\big((k,0)\to(k,0)\big)+\frac{1}{V_1}\right)\exp\big(\delta(k_0-1)\big). \qquad \square$$

Consider now, for any $k \in \mathbb{N}$, the stopping times

$$T^k = \inf\{n > 0 : X(n) \leqslant (k_0 - 1) \vee k\}, \tag{46}$$

and let τ be defined in (17) (see also Appendix B).

Lemma 15. *If $m_2 < 0$, then for any $k \geqslant 0$, there exist $\widetilde{C} > 0$ and $\widetilde{\delta} > 0$ such that*

$$\mathbb{P}_{(i,j)}\big(Y(T^k) = \ell, T^k < \tau\big) \leqslant \widetilde{C} \exp\big(-\widetilde{\delta}(i+\ell)\big), \quad \forall i \geqslant k_0,\ j \in \{0, \dots, k_0 - 1\},\ \ell \geqslant 0. \tag{47}$$

Proof. The proof of this lemma uses arguments similar to those in the proofs of Lemma 4 and Lemma 9. First, the same argument as in the proof of Lemma 4 shows that for $\alpha = (0, \theta)$ with $\theta > 0$ small enough, $R(\alpha) < 1$, see (26) and (27). Now, for any $n \geqslant 0$ and $k \geqslant 0$,

$$\begin{aligned}\mathbb{P}_{(i,j)}\big(Y(T_1^k) = \ell, T^k = n < \tau\big) &= \mathbb{P}_{(i,j)}\big(Y(n) = \ell, T^k = n < \tau\big)\\ &\leqslant \exp(-\theta\ell)\mathbb{E}_{(i,j)}\big(\exp\langle\alpha, Z(n)\rangle; T^k = n < \tau\big).\end{aligned}$$

Moreover, it follows from the definitions (17) and (46) of the stopping times τ and T_1^k that for $\alpha = (0, \theta)$ and all $n > 1$,

$$\begin{aligned}&\mathbb{E}_{(i,j)}\big(\exp\langle\alpha, Z(n)\rangle; T_1^k = n < \tau\big)\\ &\quad= \sum_{\substack{i' > (k_0-1)\vee k\\ j' > k_0}} \mathbb{P}_{(i,j)}\big(Z(1) = (i', j')\big)\mathbb{E}_{(i',j')}\big(\exp\langle\alpha, Z(n-1)\rangle; T^k = n-1 < \tau\big)\\ &\quad\leqslant \sum_{\substack{i' > (k_0-1)\vee k\\ j' > k_0}} \mathbb{P}_{(i,j)}\big(Z(1) = (i', j')\big)\mathbb{E}_{(i',j')}\big(\exp\langle\alpha, Z_0(n-1)\rangle\big)\\ &\quad\leqslant \sum_{\substack{i' > (k_0-1)\vee k\\ j' > k_0}} \mathbb{P}_{(i,j)}\big(Z(1) = (i', j')\big)\exp\big(\theta j'\big)R(\alpha)^{n-1}\\ &\quad\leqslant \mathbb{E}_{(i,j)}\big(\exp(\theta Y(1))\big)R(\alpha)^{n-1},\end{aligned}$$

with R as in (26). Moreover, using Assumption 2, we have for $\theta > 0$ small enough

$$\mathbb{E}_{(i,j)}\big(\exp(\theta Y(1))\big) \leqslant C_0.$$

Hence, for $\alpha = (0, \theta)$ with $\theta > 0$ small enough, we conclude that for all $n > 1$,

$$\mathbb{P}_{(i,j)}\big(Y(T^k) = \ell, \widehat{T}_2 = n < \tau\big) \leqslant C_0 R(\alpha)^{n-1}\exp(-\theta\ell),$$

with $R(\alpha) < 1$. Using now the same arguments as in the proof of Lemma 9, we obtain

$$\begin{aligned}\mathbb{P}_{(i,j)}\big(Y(T^k) = \ell, T^k < \tau\big) &= \sum_{n \geqslant (i-(k_0-1)\vee k)/k_0} \mathbb{P}_{(i,j)}\big(Y(T^k) = \ell, T^k = n < \tau\big)\\ &\leqslant C_0 \exp(-\theta\ell) \sum_{n \geqslant (i-(k_0-1)\vee k)/k_0} R(\alpha)^{n-1}\\ &\leqslant C_0 \exp(-\theta\ell)(1 - R(\alpha))^{-1} R(\alpha)^{(i-(k_0-1)\vee k)/k_0 - 1}.\end{aligned}$$

The last relation proves (47) with $\widetilde{\delta} = \min\{\theta, \frac{\log R(\alpha)}{k_0}\}$ and

$$\widetilde{C} = C_0(1 - R(\alpha))^{-1} R(\alpha)^{-(k_0-1\vee k)/k_0 - 1}.$$

□

Lemma 16. *For any $i, j, k \in \mathbb{N}$ and $\ell > k_0$, the following identities hold:*

$$g\big((i,j) \to (k,\ell)\big) = \widehat{g}_1^+\big((i,j) \to (k,\ell)\big) + \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j) \to (i',j')\big)\times \tag{48}$$

$$\times \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''\geqslant k_0}} \mathbb{P}_{(i',j')}\big(Z(T^k) = (i'',j''), T^k < \tau\big)\widehat{g}_1^+\big((i'',j'') \to (k,j)\big),$$

$$\mathbb{P}_{(i,j)}\big(\tau = \infty\big) = \mathbb{P}_{(i,j)}\big(\mathcal{N}_1 = 0\big), \tag{49}$$

$$\mathbb{P}_{(i,j)}\big(\tau = \infty\big) = \mathbb{P}_{(i,j)}\big(T^k < \infty, \tau = \infty\big), \tag{50}$$

and

$$\sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j) \to (i',j')\big) \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k,\\ j''\geqslant k_0}} \mathbb{P}_{(i',j')}\big(Z(T^k) = (i'',j''), T^k < \tau\big)\mathbb{P}_{(i'',j'')}\big(\tau = \infty\big)$$
$$= \mathbb{P}_{(i,j)}\big(1 \leqslant \mathcal{N}_1 < \infty\big). \tag{51}$$

Proof. To prove the identity (48) let us remark that for any $n > 0$,

$$\mathbb{P}_{(i,j)}\big(Z(n) = (k,\ell)\big) = \mathbb{P}_{(i,j)}\big(Z(n) = (k,\ell), \tau > n\big) + \mathbb{P}_{(i,j)}\big(Z(n) = (k,\ell), \tau \leqslant n\big),$$

with

$$\mathbb{P}_{(i,j)}\big(Z(n) = (k,\ell), \tau \leqslant n\big)$$
$$= \sum_{s=1}^{n} \sum_{i'\geqslant 0, 0\leqslant j'<k_0} \mathbb{P}_{(i,j)}\big(Z(s) = (i',j')\big)\mathbb{P}_{(i',j')}\big(Z(n-s) = (k,\ell), \tau > n-s\big).$$

Moreover, according to the definition of the stopping time T^k, on the event $\{Z(n) = (k,\ell)\}$, on has $T^k \leqslant n$, and consequently,

$$\mathbb{P}_{(i',j')}\big(Z(n) = (k,\ell), \tau > n\big) = \mathbb{P}_{(i',j')}\big(Z(n) = (k,\ell), \tau > n \geqslant T^k\big)$$
$$= \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k,\\ j''\geqslant k_0}} \sum_{s=1}^{n} \mathbb{P}_{(i',j')}\big(Z(s) = (i'',j''), T^k = s\big)\mathbb{P}_{(i'',j'')}\big(Z(n-s) = (k,\ell), \tau > n\big).$$

Hence, for any $i, j, k \in \mathbb{N}$ and $\ell \geqslant k_0$,

$$g\big((i,j) \to (k,\ell)\big) = \widehat{g}_1^+\big((i,j) \to (k,\ell)\big) + \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j) \to (i',j')\big)\widehat{g}_1^+\big((i',j') \to (k,j)\big),$$

with

$$\widehat{g}_1^+\big((i',j') \to (k,j)\big) = \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''\geqslant k_0}} \mathbb{P}_{(i',j')}\big(Z(T^k) = (i'',j''), T^k < \tau\big)\widehat{g}_1^+\big((i'',j'') \to (k,j)\big).$$

When combined together, the above relations prove (48).

Equation (49) follows from the definition of the stopping time τ and the variable $\mathcal{N}_1$, see (8) and (17).

The identity (50) will follow rather easily from the definition of the stopping times τ, T^k, T_1^k and τ_1. We first consider the case of $(i,j) \in \mathbb{N}^2$ with $j \geqslant k_0$. We have

$$\mathbb{P}_{(i,j)}\big(T^k < \infty, \tau = \infty\big) = \mathbb{P}_{(i,j)}\big(T_1^k < \infty, \tau_1 = \infty\big) = \mathbb{P}_{(i,j)}\big(\tau_1 = \infty\big),$$

where the last relation holds because $m_1 < 0$ and consequently $\mathbb{P}_{(i,j)}\big(T_1^k < \infty\big) = 1$. Moreover, since for $(i,j) \in \mathbb{N}^2$ with $j \geqslant k_0$

$$\mathbb{P}_{(i,j)}\big(\tau_1 = \infty\big) = \mathbb{P}_{(i,j)}\big(\tau = \infty\big),$$

we conclude therefore that for $(i,j) \in \mathbb{N}^2$ with $j \geqslant k_0$, (50) holds. We now deal with pairs $(i,j) \in \mathbb{N}^2$ satisfying $0 \leqslant j < k_0$. We then have

$$\begin{aligned}
\mathbb{P}_{(i,j)}\big(T^k < \infty, \tau = \infty\big) &= \mathbb{P}_{(i,j)}\big(T^k = 1, \tau = \infty\big) + \mathbb{P}_{(i,j)}\big(1 < T^k < \infty, \tau = \infty\big) \\
&= \sum_{\substack{0 \leqslant i' \leqslant (k_0-1)\vee k, \\ j' \geqslant k_0}} \mathbb{P}_{(i,j)}\big(Z(1) = (i',j')\big)\mathbb{P}_{(i',j')}\big(\tau = \infty\big) \\
&\qquad + \sum_{\substack{i' > (k_0-1)\vee k, \\ j' \geqslant k_0}} \mathbb{P}_{(i,j)}\big(Z(1) = (i',j')\big)\mathbb{P}_{(i',j')}\big(T^k < \infty, \tau = \infty\big) \\
&= \sum_{\substack{0 \leqslant i' \leqslant (k_0-1)\vee k, \\ j' \geqslant k_0}} \mathbb{P}_{(i,j)}\big(Z(1) = (i',j')\big)\mathbb{P}_{(i',j')}\big(\tau = \infty\big) \\
&\qquad + \sum_{\substack{i' > (k_0-1)\vee k, \\ j' \geqslant k_0}} \mathbb{P}_{(i,j)}\big(Z(1) = (i',j')\big)\mathbb{P}_{(i',j')}\big(\tau = \infty\big) \\
&= \mathbb{P}_{(i,j)}\big(T^k = 1, \tau = \infty\big) + \mathbb{P}_{(i,j)}\big(T^k > 1, \tau = \infty\big) \\
&= \mathbb{P}_{(i,j)}\big(\tau = \infty\big).
\end{aligned}$$

Relation (50) is therefore proved.

We conclude by providing the proof of (51). We start by the right-hand side of (51) and we prove that it equals the probability $\mathbb{P}_{(i,j)}\big(1 \leqslant \mathcal{N}_1 < \infty\big)$:

$$\begin{aligned}
&\sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j)\to(i',j')\big) \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''\geqslant k_0}} \mathbb{P}_{(i',j')}\big(Z(T^k)=(i'',j''), T^k<\tau\big)\mathbb{P}_{(i'',j'')}\big(\tau=\infty\big)\\
&\quad = \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j)\to(i',j')\big) \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''\geqslant k_0}} \mathbb{P}_{(i',j')}\big(Z(T^k)=(i'',j''), T^k<\infty, \tau=\infty\big)\\
&\quad = \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j)\to(i',j')\big)\mathbb{P}_{(i',j')}\big(T^k<\infty,\tau=\infty\big)\\
&\quad = \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j)\to(i',j')\big)\mathbb{P}_{(i',j')}\big(\tau=\infty\big)\\
&\quad = \sum_{n=0}^{\infty}\sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} \mathbb{P}_{(i,j)}\big(Z(n)=(i',j') \text{ and } \forall s\geqslant n,\, Y(s)\geqslant k_0\big)\\
&\quad = \sum_{n=0}^{\infty} \mathbb{P}_{(i,j)}\big(Y(n)<k_0 \text{ and } \forall s\geqslant n,\, Y(s)\geqslant k_0\big)\\
&\quad = \mathbb{P}_{(i,j)}\big(\exists n\geqslant 1 \text{ such that } Y(n)<k_0 \text{ and } \forall s\geqslant n,\, Y(s)\geqslant k_0\big)\\
&\quad = \mathbb{P}_{(i,j)}\big(1\leqslant \mathcal{N}_1<\infty\big).
\end{aligned}$$

The proof is complete. ∎

6. Proof of Theorem 1

Using all the results of Section 5, we are ready to complete the proof of Theorem 1.

Proof. Denote for $(i,j),(k,\ell)\in\mathbb{N}^2$,

$$\Delta_1\big((i,j)\to(k,\ell)\big) = \left| g\big((i,j)\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i,j)}\big(\mathcal{N}_1<\infty\big)\right|,$$
$$\Delta_1^+\big((i,j)\to(k,\ell)\big) = \left| \widehat{g}_1^+\big((i,j)\to(k,\ell)\big) - \frac{\pi_1(k)}{V_1}\mathbb{P}_{(i,j)}\big(\tau=\infty\big)\right|.$$

From Lemma 16, it follows that for any $(i,j),(k,\ell)\in\mathbb{N}^2$ with $\ell\geqslant k_0$,

$$\begin{aligned}
\Delta_1\big((i,j)\to(k,\ell)\big) \leqslant \Delta_1^+\big((i,j)\to(k,\ell)\big) + \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} g\big((i,j)\to(i',j')\big)\\
\times \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''\geqslant k_0}} \mathbb{P}_{(i',j')}\big(Z(T^k)=(i'',j''), T^k<\tau\big)\Delta_1^+\big((i'',j'')\to(k,\ell)\big).
\end{aligned}$$

Recall moreover that by Proposition 3,

$$\widetilde{C}_k = \sup_{i,j,\ell\in\mathbb{N}} g\big((i,j)\to(k,\ell)\big)<\infty, \tag{52}$$

and by Lemma 15, for any $i' \geqslant 0$, $0 \leqslant j' < k_0$, $0 \leqslant i'' \leqslant (k_0-1) \vee k$ and $j'' \geqslant k_0$,

$$\mathbb{P}_{(i',j')}\big(Z(T^k) = (i'',j''), T^k < \tau\big) \leqslant \mathbb{P}_{(i',j')}\big(Y(T^k) = j'', T^k < \tau\big) \leqslant \widetilde{C}\exp\big(-\widetilde{\delta}(i'+j'')\big).$$

Hence,

$$\begin{aligned}\Delta_1\big((i,j)\to(k,\ell)\big) &\leqslant \Delta_1^+\big((i,j)\to(k,\ell)\big)\\ &\quad + \widetilde{C}_k \sum_{\substack{0\leqslant j'<k_0\\ i'\geqslant 0}} \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''\geqslant k_0}} \widetilde{C}\exp\big(-\widetilde{\delta}(i'+j'')\big)\Delta_1^+\big((i'',j'')\to(k,\ell)\big)\\ &\leqslant \Delta_1^+\big((i,j)\to(k,\ell)\big) + A_k \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''\geqslant k_0}} \exp\big(-\widetilde{\delta}j''\big)\Delta_1^+\big((i'',j'')\to(k,\ell)\big),\end{aligned} \tag{53}$$

with $A_k = \widetilde{C}_k\widetilde{C}k_0(1-e^{-\widetilde{\delta}})^{-1}$. Recall that by Proposition 14, for any $i,j,k\in\mathbb{N}$ and $\ell\geqslant k_0$,

$$\big|\Delta_1^+\big((i,j)\to(k,\ell)\big)\big| \leqslant C_k\exp\big(-\delta_k(\ell-j)\big), \tag{54}$$

with some constants $C_k>0$ and $\delta_k>0$ not depending on $i,j,\ell\in\mathbb{N}$, and remark moreover that for any $k\in\mathbb{N}$,

$$\begin{aligned}\sup_{i,j,\ell\in\mathbb{N}} \Delta_1^+\big((i,j)\to(k,\ell)\big) &\leqslant \sup_{i,j,\ell\in\mathbb{N}} g_1^+\big((i,j)\to(k,\ell)\big) + \sup_{i,j,\ell\in\mathbb{N}} \pi_1(k)\mathbb{P}_{(i,j)}\big(\tau_1^+=\infty\big)/V_1\\ &\leqslant \sup_{i,j,\ell\in\mathbb{N}} g\big((i,j)\to(k,\ell)\big) + 1/V_1\\ &\leqslant \widetilde{C}_k + 1/V_1 < \infty.\end{aligned} \tag{55}$$

To prove Theorem 1, we split the right-hand side of (53) into two parts:

$$\Sigma_1 = A_k \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''>\ell/2}} \exp\big(-\widetilde{\delta}j''\big)\Delta_1^+\big((i'',j'')\to(k,\ell)\big),$$

$$\Sigma_2 = \Delta_1^+\big((i,j)\to(k,\ell)\big) + A_k \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ k_0<j''\leqslant\ell/2}} \exp\big(-\widetilde{\delta}j''\big)\Delta_1^+\big((i'',j'')\to(k,\ell)\big).$$

In order to estimate Σ_1, we use the upper bound (55):

$$\Sigma_1 \leqslant A_k(\widetilde{C}_k+1/V_1) \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ j''>\ell/2}} \exp\big(-\widetilde{\delta}j''\big) \leqslant A_k(\widetilde{C}_k+1/V_1)\frac{(k_0-1)\vee k+1}{1-e^{-\widetilde{\delta}}}\exp\big(-\widetilde{\delta}\ell/2\big), \tag{56}$$

and to estimate Σ_2, we use the inequality (54):

$$\begin{aligned}\Sigma_2 &\leqslant C_k\exp\big(-\delta_k(\ell-j)\big) + A_k \sum_{\substack{0\leqslant i''\leqslant (k_0-1)\vee k\\ k_0<j''\leqslant\ell/2}} \exp\big(-\widetilde{\delta}j''\big)C_k\exp(-\delta_k(\ell-j''))\\ &\leqslant C_k\exp\big(-\delta_k(\ell-j)\big) + A_kC_k\frac{(k_0-1)\vee k+1}{1-e^{-\widetilde{\delta}}}\exp\big(-\delta_k\ell/2\big).\end{aligned} \tag{57}$$

When combined together, (56) and (57) show that for any $k \in \mathbb{N}$, there exist $C'_k > 0$ and $\delta'_k > 0$ such that

$$\left| g\big((i,j) \to (k,\ell)\big) - \frac{\pi_1(k)}{V_1} \mathbb{P}_{(i,j)}(\mathcal{N}_1 < \infty) \right| = \left|\Delta\big((i,j) \to (k,\ell)\big)\right| \\ \leqslant \Sigma_1 + \Sigma_2 \leqslant C'_k \exp(-\delta'_k(\ell - j)),$$

and consequently, the first assertion of Theorem 1 holds. The proof of the second assertion of this theorem is entirely similar. □

7. Preliminary results to the proof of Theorems 2 and 3

In this section, our main objective is to introduce (mostly analytic) tools for the proof of Theorems 2 and 3, which will be provided in the following section, Section 8. Contrary to the previous Sections 5 and 6, where we use purely probabilistic arguments, we move here to an analytic framework: we introduce the generating functions of the Green functions and prove that they satisfy various functional equations, starting from which we will deduce contour integral formulas for the Green functions. Applying asymptotic techniques to these integrals will finally lead to our main results.

7.1. **Functional equations for the Green functions generating functions.** We first introduce the kernels:

$$(58) \quad \begin{cases} Q(x,y,z) &=& x^{k_0} y^{k_0} \big(z \sum_{i,j \geqslant -k_0} \mu(i,j) x^i y^j - 1\big), \\ q'_\ell(x,y,z) &=& x^{k_0} y^{\ell} \big(z \sum_{i \geqslant -k_0, j\geqslant -\ell} \mu'_\ell(i,j) x^i y^j - 1\big), & 0 \leqslant \ell \leqslant k_0 - 1, \\ q''_k(x,y,z) &=& x^{k} y^{k_0} \big(z \sum_{i \geqslant -k, j\geqslant -k_0} \mu''_k(i,j) x^i y^j - 1\big), & 0 \leqslant k \leqslant k_0 - 1, \\ q_{k,\ell}(x,y,z) &=& x^{k} y^{\ell} \big(z \sum_{i \geqslant -k, j\geqslant -\ell} \mu_{k,\ell}(i,j) x^i y^j - 1\big), & 0 \leqslant k,\ell \leqslant k_0 - 1. \end{cases}$$

Each kernel corresponds to a homogeneity domain of Figure 2. Let also the generating functions of the z-Green functions be

$$(59) \quad \begin{cases} G(x,y,z) &=& \sum_{n\geqslant 0} \sum_{i,j\geqslant k_0} \mathbb{P}_{(i_0,j_0)}\big(Z(n) = (i,j)\big) x^{i-k_0} y^{j-k_0} z^n, \\ g_\ell(x,z) &=& \sum_{n\geqslant 0} \sum_{i\geqslant k_0} \mathbb{P}_{(i_0,j_0)}\big(Z(n) = (i,\ell)\big) x^{i-k_0} z^n, & 0 \leqslant \ell \leqslant k_0 - 1, \\ \widetilde{g}_k(y,z) &=& \sum_{n\geqslant 0} \sum_{j\geqslant k_0} \mathbb{P}_{(i_0,j_0)}\big(Z(n) = (k,j)\big) y^{j-k_0} z^n, & 0 \leqslant k \leqslant k_0 - 1, \\ f_{i_0,j_0}(x,y,z) &=& \sum_{0\leqslant k,\ell \leqslant k_0-1} \big(\sum_{n\geqslant 0} \mathbb{P}_{(i_0,j_0)}\big(Z(n) = (k,\ell)\big) z^n\big) q_{k,\ell}(x,y,z) + x^{i_0} y^{j_0}. \end{cases}$$

They are all well defined when $|x| < 1$, $|y| < 1$ and $|z| < 1$.

Lemma 17. *The following equation holds true, for any $|x| < 1$, $|y| < 1$ and $|z| < 1$:*

$$(60) \quad -Q(x,y,z)G(x,y,z) = \sum_{\ell=0}^{k_0-1} q'_\ell(x,y,z) g_\ell(x,z) + \sum_{k=0}^{k_0-1} q''_k(x,y,z)\widetilde{g}_k(y,z) + f_{i_0,j_0}(x,y,z).$$

Proof. The generating functions are clearly convergent for x, y, z less than 1 in modulus, as the coefficients $\mathbb{P}_{(i_0,j_0)}\big(Z(n) = (i,j)\big) = p^{(n)}\big((i_0,j_0) \to (i,j)\big)$ are smaller than 1. In the case of nearest-neighbor random walks, the functional equation (60) has been obtained in [25, Lem. 3.16]. All generating functions remain convergent in the case of larger steps, and the proof of (60) follows exactly the same line in this generalized framework. □

Let now

$$Q(x,y) = Q(x,y,1),\ q'_\ell(x,y) = q'_\ell(x,y,1),\ q''_k(x,y) = q''_k(x,y,1),\ q_{k,\ell}(x,y) = q_{k,\ell}(x,y,1)$$

be the evaluations of the kernels (58) at $z=1$. Finally, the generating functions of the Green functions are

$$\tag{61} \left\{ \begin{array}{rcll} G(x,y) &=& \sum_{i,j\geqslant k_0} g\big((i_0,j_0)\to(i,j)\big)x^{i-k_0}y^{j-k_0}, & \\ g_\ell(x) &=& \sum_{i\geqslant k_0} g\big((i_0,j_0)\to(i,\ell)\big)x^{i-k_0}, & 0\leqslant \ell\leqslant k_0-1, \\ \widetilde{g}_k(y) &=& \sum_{j\geqslant k_0} g\big((i_0,j_0)\to(k,j)\big)y^{j-k_0}, & 0\leqslant k\leqslant k_0-1, \\ f_{i_0,j_0}(x,y) &=& \sum_{0\leqslant k,\ell\leqslant k_0-1} g\big((i_0,j_0)\to(k,\ell)\big)q_{k,\ell}(x,y) + x^{i_0}y^{j_0}. & \end{array} \right.$$

The generating functions g_ℓ and $\widetilde{g}_k$ in (61) are strongly related to $\mathcal{G}_{(i,j)\to(\cdot,\ell)}$ and $\mathcal{G}_{(i,j)\to(k,\cdot)}$ in (82) and (83). More specifically, they just differ by polynomial terms; for example,

$$\tag{62} \mathcal{G}_{(i,j)\to(k,\cdot)}(y) = y^{k_0}\widetilde{g}_k(y) + \sum_{0\leqslant j<k_0} g\big((i_0,j_0)\to(k,j)\big)y^j.$$

Lemma 18. *All generating functions* (61) *are absolutely convergent on the bidisk* $\{(x,y)\in\mathbb{C}^2 : |x|<1, |y|<1\}$, *where they satisfy the functional equation:*

$$\tag{63} -Q(x,y)G(x,y) = \sum_{\ell=0}^{k_0-1} q'_\ell(x,y)g_\ell(x) + \sum_{k=0}^{k_0-1} q''_k(x,y)\widetilde{g}_k(y) + f_{i_0,j_0}(x,y).$$

Although Equation (63) appears formally as the evaluation of (60) at $z=1$, its proof needs some care, as it is not at all clear a priori that the generating function $G(x,y)$ converges for $|x|<1$ and $|y|<1$. (This convergence actually constitutes a first non-obvious estimate.)

Proof of Lemma 18. The series $\sum_{n\geqslant 0} p^{(n)}\big((i_0,j_0)\to(k,\ell)\big)$ is convergent to $g\big((i_0,j_0)\to(k,\ell)\big)$, and by Proposition 3,

$$\sup_{\substack{i\geqslant k_0\\ \ell\in\{0,\ldots,k_0-1\}}} g\big((i_0,j_0)\to(i,\ell)\big)<\infty \quad\text{and}\quad \sup_{\substack{j\geqslant k_0\\ k\in\{0,\ldots,k_0-1\}}} g\big((i_0,j_0)\to(k,j)\big)<\infty.$$

Accordingly, the series $g_\ell(x,1)$ and $\widetilde{g}_k(1,y)$ are absolutely convergent respectively for any x and y with $|x|<1$ and $|y|<1$. Then, by Abel's theorem on power series, the limit of the right-hand side of (60) as $z\to 1$ exists and equals the right-hand side of (63).

As a consequence, the limit of the left-hand side of (60) does exist as well, so that $\lim_{z\to1} G(x,y,z)$ exists for any pair (x,y) in the set

$$\tag{64} \{(x,y)\in\mathbb{C}^2 : |x|<1, |y|<1, Q(x,y,1)\neq 0\}.$$

Furthermore, for any $(x,y)\in(0,1)^2$, the series $G(x,y,z)$ of z has real non-negative coefficients. It follows that $G(x,y,1)$ converges for any real (x,y) from the set (64), which is dense in $(0,1)^2$. Therefore $G(x,y,1)$ is absolutely convergent for any complex (x,y) with $|x|,|y|<1$ and again by Abel's theorem, $\lim_{z\to1} G(x,y,z) = G(x,y,1)$. Hence the limit of the left-hand side of (60) as $z\to1$ exists and equals (63). The equation (63) is proved. $\square$

7.2. **Preliminary results on the zero-set $Q(x,y)=0$.** We gather in a single lemma several statements on the zeros of the two-variable polynomial $Q(x,y)$ and on the one-variable polynomials $Q(x,1)$ and $Q(1,y)$. The last ones may be interpreted as kernels of one-dimensional random walks (and a large literature exists around this type of models). These statements will be used in the proof of Theorem 2.

Although, under our standing assumptions, the equation $y \mapsto Q(x,y)=0$ (x being fixed) has in general an infinite number of solutions, two roots play a very special role and carry out most of the probabilistic information about the model. They will be denoted Y_0 and Y_1.

Lemma 19. *The following assertions hold:*

(i) *For $|x|=1$ and $|y|=1$ with $(x,y)\neq(1,1)$, we have $Q(x,y)\neq 0$.*

(ii) *There exists a neighborhood $O_\delta(1)=\{x\in\mathbb{C}:|x-1|<\delta\}$ of 1, with $\delta>0$ small enough, inside of which there exists a unique function Y_0 analytic in $O_\delta(1)$, which satisfies $Y_0(1)=1$ and $Q(x,Y_0(x))=0$ for all $x\in O_\delta(1)$. The function Y_0 is one-to-one from $O_\delta(1)$ onto $U_\delta(1)=Y_0(O_\delta(1))$, which is a neighborhood of 1.*

(iii) *For $\varepsilon>0$ small enough, $Y_0(1+\varepsilon)<1$.*

(iv) *There exists a domain (see Figure 6)*

$$V=\{(1-\varepsilon)+\varepsilon e^{i\phi}:\varepsilon\in(0,\varepsilon_0],\phi\in[-\phi_0,\phi_0]\} \tag{65}$$

such that $V\subset O_\delta(1)$, and for any $x\in V$, we have $|x|<1$ and $|Y_0(x)|<1$.

(v) *On $(0,\infty)$, the function $Q(1,y)$ admits exactly two zeros, at $y_0=1$ and $y_1>1$. Moreover, for any y with $1<|y|<y_1$, $Q(1,y)\neq 0$.*

(vi) *For small $\varepsilon>0$, the function $Q(1+\varepsilon,y)$ has exactly two zeros on $(0,\infty)$. One of them is $Y_0(1+\varepsilon)$, as defined in (ii) and (iii). The other zero is called $Y_1(1+\varepsilon)$. We have $Y_0(1+\varepsilon)<Y_1(1+\varepsilon)$. Furthermore, for any y with $Y_0(1+\varepsilon)<|y|<Y_1(1+\varepsilon)$, we have $Q(1+\varepsilon,y)\neq 0$.*

(vii) *For $\varepsilon>0$ small enough, any x with $|x|=1+\varepsilon$ and any y with $Y_0(1+\varepsilon)<|y|<Y_1(1+\varepsilon)$, we have $Q(1+\varepsilon,y)\neq 0$.*

(viii) *For any $\varepsilon>0$ small enough and any $t>0$,*

$$(1+\varepsilon)Y_1(1+\varepsilon)^t>\min\{x_1,y_1^t\},$$

where x_1 is defined symmetrically as y_1 in (v), but in the x-variable.

As stated, Lemma 19 concerns the functions $Q(1,y)$, Y_0 and Y_1. Obviously, symmetric statements hold for $Q(x,1)$, X_0 and X_1.

Proof of Lemma 19. Item (i) will be the topic of the separate Lemma 20, so we start with the proof of (ii). It readily follows from the analytic implicit function theorem applied to the function $(x,y)\mapsto Q(x,y)$ in the neighborhood of $(1,1)$, noticing that $Q(1,1)=0$ and $\partial_2 Q(1,1)=\sum_{i,j} j\mu(i,j)<0$ (by assumption). Similarly, (iii) follows from the (real version of) the implicit function theorem, using once again that $\partial_2 Q(1,1)<0$.

Before starting the proof of (iv), we need the preliminary series expansion (67) below, which describes the behaviour of the modulus of $Y_0(x)$ as x lies on a circle tangent to

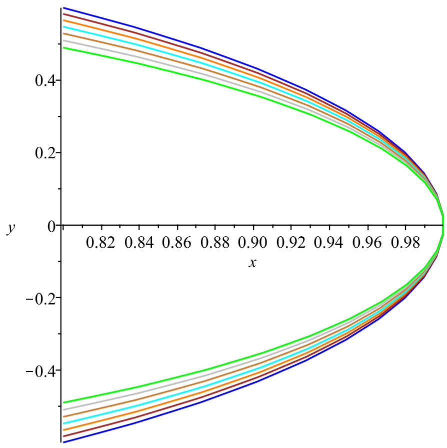

FIGURE 6. The unit circle is in blue (the rightmost curve), and the other circles are tangent to it at 1. These domains appear in the proof of item (iv) of Lemma 19.

the unit circle, but with a smaller radius (see Figure 6). First of all, take the notation $a = Y_0'(1)$ and $2b = Y_0''(1)$. Then obviously

$$Y_0(x) = 1 + a(x-1) + b(x-1)^2 + o(x-1)^2, \quad x \to 1. \tag{66}$$

A few standard computations yield that for any $\varepsilon \in [0,1]$, one has

$$|Y_0(\varepsilon + (1-\varepsilon)e^{i\phi})| = 1 + \frac{1-\varepsilon}{2}(a^2 - a - 2b + \varepsilon(2b - a^2))\phi^2 + o(\phi^2), \quad \phi \to 0. \tag{67}$$

The values of a and b are computed below, in Lemma 21, in terms of the first and second moments of the distribution μ.

For later use, let us first show that $a^2 - a - 2b < 0$. Using the explicit expressions for a and b given in Lemma 21,

$$a^2 - a - 2b = Y_0'(1)^2 - Y_0'(1) - Y_0''(1) = \frac{(\mathbb{E}X)^2\mathbb{E}(Y^2) - 2\mathbb{E}X\mathbb{E}(XY)\mathbb{E}Y + \mathbb{E}(X^2)(\mathbb{E}Y)^2}{(\mathbb{E}Y)^3}. \tag{68}$$

Since $\mathbb{E}Y < 0$, the denominator of (68) is negative, and it is enough to prove that the numerator of (68) is positive, namely,

$$(\mathbb{E}X)^2\mathbb{E}(Y^2) - 2\mathbb{E}X\mathbb{E}XY\mathbb{E}Y + \mathbb{E}(X^2)(\mathbb{E}Y)^2 > 0. \tag{69}$$

As it turns out, the above inequality is a straightforward consequence of Cauchy-Schwarz inequality applied to the random variables $\frac{X}{\mathbb{E}X}$ and $\frac{Y}{\mathbb{E}Y}$.

In order to construct the neighbourhood V in (65), we shall use the estimate (67), as follows. Let us first observe that for $\varepsilon \in [0,1]$ small enough (say $\varepsilon \in [0,\varepsilon_0]$), one has $a^2 - a - 2b + \varepsilon(2b - a^2) < 0$ (indeed, $a^2 - a - 2b < 0$, see above). One can also make the term $o(\phi^2)$ in (67) uniform in $\varepsilon \in [0,\varepsilon_0]$, as the point $x = 1$ is regular for the function $Y_0(x)$ (and its derivatives). It follows that there exists a value $\phi_0 > 0$ such that for all $\phi \in (-\phi_0,\phi_0) \setminus \{0\}$ and all $\varepsilon \in [0,\varepsilon_0]$, $|Y_0(\varepsilon + (1-\varepsilon)e^{i\phi})| < 1$. In conclusion, the neighbourhood V may be taken as the union of all these small arcs of circle as in (65) (see also Figure 6). Item (iv) is proved.

We now prove (v). We have $Q(1,y) = y^{k_0}(P(y)-1)$, where

$$P(y) = \sum_{j=-k_0}^{\infty} \mu(-,j)y^j. \tag{70}$$

By our main assumptions, the series $Q(1,y)$ has a radius of convergence $R \in (1,\infty]$. The function $P(y)$ in (70) is well defined on $(0,R)$ and is strictly convex. Furthermore, one has $\lim_{y\to 0+} P(y) = +\infty$. There exists a unique $\tau \in (0,R)$ such that P is (strictly) decreasing on $(0,\tau)$ and (strictly) increasing on (τ,R); τ is called the structural constant, and $\rho = 1/P(\tau)$ is called the structural radius (see [2, Lem. 1]). One has $P'(\tau) = 0$. In case of a negative drift, one has $\tau > 1$, since

$$P'(1) = \sum_{j=-k_0}^{\infty} j\mu(-,j) = \sum_{i,j=-k_0}^{\infty} j\mu(i,j) < 0.$$

By Cramer's condition, one has $Q(1,R) \in (0,\infty]$, so that there exists $y_1 \in (\tau,R)$ such that $P(y_1) = 0$. Furthermore, $P(y) \in (0,1)$ for any $y \in (1,y_1)$.

A general fact about Laurent polynomials with non-negative coefficients enters the game: $|P(y)| \leqslant P(|y|)$. The inequalities $P(y) \in (0,1)$ for any $y \in (1,y_1)$ thus imply that $|P(y)| < 1$ for all $1 < |y| < y_1$.

We pursue by showing (vi). We proceed as in the proof of (v). Using (58), we first rewrite the equality $Q(1+\varepsilon,y) = 0$ as

$$P_\varepsilon(y) = 1, \quad \text{where } P_\varepsilon(y) = \sum_{j=-k_0}^{\infty}\left(\sum_{i=-k_0}^{\infty} \mu(i,j)(1+\varepsilon)^i\right) y^j.$$

The polynomial P_ε above has non-negative coefficients and is strictly convex on $(0,R_\varepsilon)$, where R_ε is the radius of convergence of P_ε. For $\varepsilon = 0$, it is equal to P as defined in (70). Using that $P'(1) < 0$, we deduce that for $\varepsilon > 0$ small enough, $P_\varepsilon'(Y_0(1+\varepsilon)) < 0$. Then for $y \in (Y_0(1+\varepsilon), Y_1(1+\varepsilon))$, P_ε takes values in $(0,1)$. We conclude as in (v).

In order to prove (vii), we first write, using (58), that $Q(x,y) = x^{k_0}y^{k_0}(\mu(x,y)-1)$. Then, by positivity of the coefficients,

$$|\mu(x,y)| \leqslant \mu(|x|,|y|),$$

and we conclude using (vi).

It remains to prove (viii). Consider the function $g_t(x,y) = xy^t$ on $Q(x,y) = 0$ with $(x,y) \in (0,\infty)^2$ and look for its extrema. Equivalently, look at the extrema of $(u,v) \mapsto e^{u+tv}$ on

$$\mathcal{Q} = \{(u,v) \in \mathbb{R}^2 : \sum_{i,j} \mu(i,j)e^{iu+jv} = 1\} = \{(u,v) \in \mathbb{R}^2 : Q(e^u,e^v) = 0\}, \tag{71}$$

see Figure 7. There are three particular points on the latter curve, namely, $(0,0)$, $(u_0,0) = (\log x_1, \log 1)$ and $(0,v_0) = (\log 1, \log y_1)$.

Let us show that on

$$\mathcal{P} = \mathcal{Q} \cap (0,\infty)^2,$$

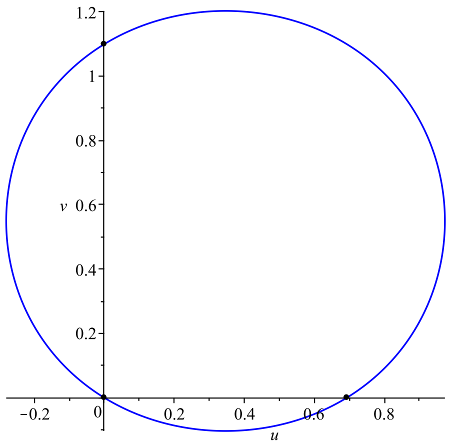

FIGURE 7. An example of domain $\mathcal{Q}$ as in (71), for the random walk with transition probabilities defined by (12). The points $(0,0)$, $(u_0,0) = (\log 2, 0)$ and $(0,v_0) = (0,\log 3)$ are represented with bullets.

which is the part of $\mathcal{Q}$ between $(u_0,0)$ and $(0,v_0)$ run counterclockwise, the function $u+tv$ is strictly bigger than its values at the boundary points:

$$u + tv > \min\{u_0, tv_0\}, \quad \forall (u,v) \in \mathcal{P},\ \forall t \in (0,\infty). \tag{72}$$

Consider the critical points of $u+tv$ on $\mathcal{Q}$. A necessary condition is that $u'(v) = -t$. But $u'(v) = -\frac{\partial_v Q(e^u,e^v)}{\partial_u Q(e^u,e^v)}$, so that at critical points one must have $t = \frac{\partial_v Q(e^u,e^v)}{\partial_u Q(e^u,e^v)}$. On the other hand, it has been established by Hennequin [18] that the mapping

$$(u,v) \mapsto \frac{\operatorname{grad} Q(e^u,e^v)}{\|\operatorname{grad} Q(e^u,e^v)\|}$$

is a diffeomorphism between $\mathcal{Q}$ and the unit sphere. Then the critical points (u,v) are the images of the points of the unit sphere such that the ratio of the second coordinate by the first coordinate equals t. There are exactly two points on the unit sphere with this property, so that there exist exactly two critical points of $u+tv$ on $\mathcal{Q}$. The function $u+tv$ being continuous on $\mathcal{Q}$, it reaches its maximum and minimum. Then one of these points must be its minimum on $\mathcal{Q}$ and cannot belong to $\mathcal{P}$, since the function is positive on this part, while it vanishes at $(0,0) \in \mathcal{Q}$. The second critical point must be the maximum of $u+tv$ on $\mathcal{Q}$ and may belong to $\mathcal{P}$ or not. Furthermore, the function $u+tv$ must be strictly monotonous on $\mathcal{Q}$ between these two critical points. Hence the estimate (72) holds. It implies

$$\log(1+\varepsilon) + t\log Y_1(1+\varepsilon) > \min\{\log x_1, t\log y_1\},$$

which proves (viii). □

Lemma 20. *Let $(\mu(m,n))_{m,n\geqslant -k_0}$ be a family of non-negative real numbers summing to one, such that the semigroup of $\mathbb{Z}^2$ generated by the support $\{(m,n)\in\mathbb{Z}^2 : \mu(m,n)\neq 0\}$ is $\mathbb{Z}^2$ itself. If a pair $(x,y)\in\mathbb{C}^2$ with $|x|=|y|=1$ satisfies*

$$\mu(x,y) = \sum_{m,n\geqslant -k_0} \mu(m,n)x^m y^n = 1,$$

then necessarily $x = y = 1$.

Remark that the hypothesis on the semigroup is equivalent to the irreducibility of the random walk Z_0 on $\mathbb{Z}^2$ whose increment distribution is given by μ. In dimension 1, a proof may be found in [2, Eq. (35)]. The proof of Lemma 20 is very standard and will be omitted.

Lemma 21. *Let* (X, Y) *be a random vector with distribution* μ. *One has*

$$Y_0'(1) = -\frac{\mathbb{E}X}{\mathbb{E}Y} \tag{73}$$

and

$$Y_0''(1) = \frac{(\mathbb{E}X)^2\mathbb{E}Y - (\mathbb{E}X)^2\mathbb{E}(Y^2) + 2\mathbb{E}X\mathbb{E}XY\mathbb{E}Y + \mathbb{E}X(\mathbb{E}Y)^2 - \mathbb{E}(X^2)(\mathbb{E}Y)^2}{2(\mathbb{E}Y)^3}. \tag{74}$$

Similarly, we could compute the derivatives of Y_1 at 1 (see Lemma 19 (vi) for the definition of Y_1) in terms of the moments of a Doob transform of (X, Y).

Proof of Lemma 21. Differentiating the identity $Q(x, Y_0(x)) = 0$, one obtains

$$\partial_1 Q(x, Y_0(x)) + Y_0'(x)\partial_2 Q(x, Y_0(x)) = 0, \tag{75}$$

and in particular (using that $Y_0(1) = 1$)

$$Y_0'(1) = -\frac{\partial_1 Q(1,1)}{\partial_2 Q(1,1)} = -\frac{\sum_{i,j} i\mu(i,j)}{\sum_{i,j} j\mu(i,j)},$$

which proves (73). Differentiating now (75), we get

$$\partial^2_{1,1}Q(x, Y_0(x)) + 2Y_0'(x)\partial^2_{1,2}Q(x, Y_0(x)) + Y_0''(x)\partial_2 Q(x, Y_0(x)) \\ + (Y_0'(x))^2\partial^2_{2,2}Q(x, Y_0(x)) = 0. \tag{76}$$

Moreover, one easily computes

$$\left\{\begin{array}{lcl} \partial^2_{1,1}Q(1,1) & = & (2k_0 - 1)\mathbb{E}X + \mathbb{E}(X^2), \\ \partial^2_{2,2}Q(1,1) & = & (2k_0 - 1)\mathbb{E}Y + \mathbb{E}(Y^2), \\ \partial^2_{1,2}Q(1,1) & = & k_0\mathbb{E}X + k_0\mathbb{E}Y + \mathbb{E}(XY). \end{array}\right. \tag{77}$$

Plugging (77) in (76) evaluated at $x = 1$, we conclude that (74) holds. $\square$

7.3. **One-dimensional stationary probabilities.** In the forthcoming proof of Theorem 2, we need to identify the invariant measure of the stationary Markov chain X_1 defined in (4), which is a one-dimensional reflected random walk on $\mathbb{N}$. The transitions of X_1 are as follows: for any $k, \ell \in \mathbb{N}$,

$$\mathbb{P}_k\big(X_1(1) = \ell\big) = \left\{\begin{array}{ll} \sum_{j\geqslant -k_0} \mu(\ell - k, j) & \text{if } k \geqslant k_0, \\ \sum_{j\geqslant -k_0} \mu_k''(\ell - k, j) & \text{if } 0 \leqslant k < k_0. \end{array}\right.$$

Using our notation (58), the associated kernels are $Q(x, 1)$ (in the regime when $k \geqslant k_0$) and $q_k''(x, 1)$ (when $0 \leqslant k < k_0$).

Lemma 22. *The invariant measure $\{\pi_1(i)\}_{i\geqslant 0}$ of X_1 can be computed as*

$$\pi_1(i) = \frac{1}{2\pi i}\int_{|x|=1-\varepsilon} \frac{\sum_{k=0}^{k_0-1}\pi_1(k)q''_k(x,1)}{x^{i-k_0+1}Q(x,1)}dx = \frac{1}{2\pi i}\int_{|x|=1+\varepsilon} \frac{\sum_{k=0}^{k_0-1}\pi_1(k)q''_k(x,1)}{x^{i-k_0+1}Q(x,1)}dx. \tag{78}$$

As $i\to\infty$, it admits the asymptotics

$$\pi_1(i) \sim \frac{c}{x_1^i}, \tag{79}$$

where the constant c is positive, and equal to

$$c = \frac{\sum_{k=0}^{k_0-1}\pi_1(k)q''_k(x_1,1)}{x_1^{-k_0+1}\partial_1 Q(x_1,1)}. \tag{80}$$

Although Lemma 22 is classical in the probabilistic literature, we present some elements of proof below, in order to make our article self-contained. We thank Onno Boxma and Dmitry Korshunov for useful bibliographic advice.

Sketch of the proof of Lemma 22. Introduce the stationary distribution generating function $\Pi_1(x) = \sum_{k=k_0}^{\infty}\pi_1(k)x^{k-k_0}$. Then the following functional equation holds (it is equivalent to the equilibrium equations):

$$Q(x,1)\Pi_1(x) = \sum_{k=0}^{k_0-1}\pi_1(k)q''_k(x,1).$$

The first integral expression in (78) (over $|x| = 1-\varepsilon$) immediately follows. Since $Q(x,1) = 0$ and $\Pi_1(1) = 1$, the right-hand side of the above identity is zero at $x = 1$ and the function Π_1 is analytic in a neighborhood of 1. We deduce the second integral representation in (78) (over $|x| = 1+\varepsilon$). Using the functional equation and the fact that $Q(x,1)$ has a simple pole at x_1 (see Lemma 19 (v)), one immediately deduces the asymptotics (79), with the expression of the constant c as in (80).

On the other hand, the (strict) positivity of the constant c in (79) is more difficult to establish (and is not clear at all from the algebraic expression of c given in (80), as for $k\geqslant 1$, $q''_k(x_1,1)$ may be negative). However, Theorem 2 in [9] shows the positivity of c for a more general class of random walks; see also [5]. □

8. Proof of Theorems 2 and 3

Let us first summarize the proof Theorem 2 in several important steps, to which we shall refer in the extended proof. First of all, it follows from the main functional equation (63) that for any $\varepsilon>0$ small enough,

$$g\big((i_0,j_0)\to(i,j)\big) = \frac{1}{(2\pi i)^2}\iint_{|x|=|y|=1-\varepsilon} \frac{\sum_{\ell=0}^{k_0-1}q'_\ell(x,y)g_\ell(x) + \sum_{k=0}^{k_0-1}q''_k(x,y)\widetilde{g}_k(y) + f_{i_0,j_0}(x,y)}{x^{i+k_0-1}y^{j+k_0-1}Q(x,y)}dydx. \tag{81}$$

Then:

1. We shall apply the residue theorem to the inner integral above (in y), so as to split $g\big((i_0,j_0)\to(i,j)\big)$ into two terms, see (85).

2. The first term in the decomposition (85) is a one-variable integral, to which we apply the classical residue theorem. Some technical work is needed here to prove that there is only one contributing pole, at 1 (we use several properties proved in Lemma 19).
3. The second term in (85) is a double integral over $|x| = 1-\varepsilon$ and $|y| = 1+\varepsilon$. We will again apply the residue theorem to the inner integral and, in this way, obtain a further two-term decomposition and the expression (87) of the Green function.
4. The second term in the sum (87) is studied via the residue theorem, in a similar way as the first term in the decomposition (85).
5. The third term in (87) is an integral on $|x| = |y| = 1+\varepsilon$ and is shown to be negligible.
6. Conclusion.

Before embarking in the proof, we state an equivalent, but more analytic version of Theorem 1. To that purpose, similarly to (35), we introduce the following generating functions, for respectively fixed $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$:

$$\mathcal{G}_{(i,j)\to(k,\cdot)}(x) = \sum_{\ell=0}^{\infty} g\big((i,j)\to(k,\ell)\big)x^{\ell}, \tag{82}$$

$$\mathcal{G}_{(i,j)\to(\cdot,\ell)}(x) = \sum_{k=0}^{\infty} g\big((i,j)\to(k,\ell)\big)x^{k}. \tag{83}$$

Corollary 23. *Under the Assumptions 1–5, there exists $\varepsilon > 0$ such that for any $(i,j) \in \mathbb{N}^2$ and any $k,\ell \in \mathbb{N}$, the generating functions $\mathcal{G}_{(i,j)\to(k,\cdot)}$ and $\mathcal{G}_{(i,j)\to(\cdot,\ell)}$ can be continued in a meromorphic manner in the disk $\{x \in \mathbb{C} : |x| < 1+\varepsilon\}$, with a unique simple pole, which is located at the point $x = 1$ and admits the residue*

$$\begin{aligned} \operatorname{Res}_1 \mathcal{G}_{(i,j)\to(k,\cdot)} &= \pi_1(k)\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1 < \infty\big)/V_1, \\ \operatorname{Res}_1 \mathcal{G}_{(i,j)\to(\cdot,\ell)} &= \pi_2(\ell)\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2 < \infty\big)/V_2. \end{aligned} \tag{84}$$

Proof of Theorem 2. We start with *Step 1*. Let us fix x on the circle $|x| = 1-\varepsilon$. Since the integrand of the inner integral in (81) may be continued as a meromorphic function to the larger disc $\{|y| < 1+\varepsilon\}$, see Corollary 23, we write the Green function (81) as

(85)
$$\frac{1}{2\pi i}\int_{|x|=1-\varepsilon} \sum_{y:1-\varepsilon<|y|<1+\varepsilon} \operatorname{Res} \frac{\sum_{\ell=0}^{k_0-1} q'_\ell(x,y)g_\ell(x) + \sum_{k=0}^{k_0-1} q''_k(x,y)\widetilde{g}_k(y) + f_{i_0,j_0}(x,y)}{x^{i+k_0-1}y^{j+k_0-1}Q(x,y)}dx$$
$$+\frac{1}{(2\pi i)^2}\iint_{\substack{|x|=1-\varepsilon\\|y|=1+\varepsilon}} \frac{\sum_{\ell=0}^{k_0-1} q'_\ell(x,y)g_\ell(x) + \sum_{k=0}^{k_0-1} q''_k(x,y)\widetilde{g}_k(y) + f_{i_0,j_0}(x,y)}{x^{i+k_0-1}y^{j+k_0-1}Q(x,y)}dydx.$$

Step 2 consists in studying the first integral in (85). Let us look at the residues appearing in the integrand. Obviously, the poles will be found among the zeros of $Q(x,y)$ and the poles of the numerator, x being fixed on the circle $|x| = 1-\varepsilon$.

Let $O_\delta(1)$ and $U_\delta(1)$ be the neighborhoods of 1 introduced in Lemma 19 (ii). We are going to study successively three cases (remind that, in addition, we always have $|x| = 1-\varepsilon$ and $1-\varepsilon < |y| < 1+\varepsilon$):

1. $x \notin O_\delta(1)$;
2. $x \in O_\delta(1)$ and $y \notin U_\delta(1)$;
3. $x \in O_\delta(1)$ and $y \in U_\delta(1)$.

We first consider case 1 and prove that no point will not contribute to the computation of the residues. By Lemma 19 (i), for any $|x| = |y| = 1$ with $x \notin O_\delta(1)$, the continuous function $Q(x, y)$ is non-zero. By continuity, we also have $Q(x, y) \neq 0$ for any $1-\varepsilon < |x| < 1+\varepsilon$, $1-\varepsilon < |y| < 1+\varepsilon$ with $x \notin O_\delta(1)$. Case 2 is handled symmetrically.

In case 3, then using Lemma 19 (ii), there is only one potential zero of $Q(x, y)$, namely $Y_0(x)$. We take δ sufficiently small to ensure that for all $\ell = 0, \ldots, k_0-1$, $g_\ell(x)(1-x)$ and $\widetilde{g}_k(Y_0(x))(1-Y_0(x))$ are analytic in $O_\delta(1)$, see Corollary 23 and Lemma 19 (ii), and we show that $Y_0(x)$ is not a pole. Our key argument is that $y = Y_0(x)$ will also be a zero of the numerator, and so a removable singularity of the integrand for any $x \in O_\delta(1) \setminus \{x = 1\}$.

Let us introduce the domain V as in (65) (see Lemma 19 (iv)). Since $|x| < 1$ and $|Y_0(x)| < 1$ on this set, the main equation (63) implies

$$\sum_{\ell=0}^{k_0-1} q'_\ell(x, Y_0(x)) g_\ell(x) + \sum_{k=0}^{k_0-1} q''_k(x, Y_0(x)) \widetilde{g}_k(Y_0(x)) + f_{i_0,j_0}(x, Y_0(x)) = 0, \quad \forall x \in V. \tag{86}$$

Furthermore, the left-hand side of (86) multiplied by the factor $(1-x)(1-Y_0(x))$ is an analytic function in $O_\delta(1)$, which equals zero in the domain $V \subset O_\delta(1)$. Then, by the principle of analytic continuation, the left-hand side of (86) multiplied by $(1-x)(1-Y_0(x))$ equals zero in the whole of $O_\delta(1)$. Hence the left-hand side of (86) is equal to zero in $O_\delta(1) \setminus \{x = 1\}$.

In the first integral in (85), it remains to compute the residues at the poles of the numerator. By Corollary 23, there exists only one pole of the numerator, namely, $y = 1$, which is a pole of $\widetilde{g}_k(y)$ for all $k = 0, \ldots, k_0-1$ (by (62), $\widetilde{g}_k$ and $\mathcal{G}_{(i,j)\to(k,\cdot)}$ have the same residue at 1, namely $\pi_1(k)\mathbb{P}_{(i_0,j_0)}(\mathcal{N}_1 < \infty)/V_1$). Thus we get

$$\begin{aligned} g\big((i_0, j_0) \to (i, j)\big) &= \frac{\mathbb{P}_{(i_0,j_0)}(\mathcal{N}_1 < \infty)}{V_1} \frac{1}{2\pi i} \int_{|x|=1-\varepsilon} \frac{\sum_{k=0}^{k_0-1} \pi_1(k) q''_k(x, 1)}{x^{i-k_0+1} Q(x, 1)} dx \\ &\quad + \frac{1}{(2\pi i)^2} \iint_{\substack{|x|=1-\varepsilon \\ |y|=1+\varepsilon}} \frac{\sum_{\ell=0}^{k_0-1} q'_\ell(x, y) g_\ell(x) + \sum_{k=0}^{k_0-1} q''_k(x, y) \widetilde{g}_k(y) + f_{i_0,j_0}(x, y)}{x^{i+k_0-1} y^{j+k_0-1} Q(x, y)} dx dy, \end{aligned}$$

where we inverted the order of integration in the second term. As proved in Lemma 22, the first term is the integral of an analytic function in the annulus $\{1-\varepsilon < |x| < 1+\varepsilon\}$, so that it equals the same integral over $\{|x| = 1+\varepsilon\}$, which is nothing else but the invariant measure announced in the theorem, see (78).

Step 3. We proceed with the second term of (85) as previously:

$$g\big((i_0,j_0)\to(i,j)\big)=\frac{\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)}{V_1}\pi_1(i) \tag{87}$$
$$+\frac{1}{(2\pi i)^2}\int\limits_{|y|=1+\varepsilon}\sum_{x:1-\varepsilon<|x|<1+\varepsilon}\operatorname{Res}\frac{\sum_{\ell=0}^{k_0-1}q'_\ell(x,y)g_\ell(x)+\sum_{k=0}^{k_0-1}q''_k(x,y)\widetilde{g}_k(y)+f_{i_0,j_0}(x,y)}{x^{i+k_0-1}y^{j+k_0-1}Q(x,y)}dy$$
$$+\frac{1}{(2\pi i)^2}\iint\limits_{|x|=|y|=1+\varepsilon}\frac{\sum_{\ell=0}^{k_0-1}q'_\ell(x,y)g_\ell(x)+\sum_{k=0}^{k_0-1}q''_k(x,y)\widetilde{g}_k(y)+f_{i_0,j_0}(x,y)}{x^{i+k_0-1}y^{j+k_0-1}Q(x,y)}dxdy.$$

Step 4. Using a symmetric reasoning as in Step 2, we obtain

$$g\big((i_0,j_0)\to(i,j)\big)=\frac{\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)}{V_1}\pi_1(i)+\frac{\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)}{V_2}\pi_2(j) \tag{88}$$
$$+\frac{1}{(2\pi i)^2}\iint\limits_{|x|=|y|=1+\varepsilon}\frac{\sum_{\ell=0}^{k_0-1}q'_\ell(x,y)g_\ell(x)+\sum_{k=0}^{k_0-1}q''_k(x,y)\widetilde{g}_k(y)+f_{i_0,j_0}(x,y)}{x^{i+k_0-1}y^{j+k_0-1}Q(x,y)}dxdy.$$

Step 5. Let $i,j\to\infty$. We prove that the last integral above is $o(x_1^{-i}+y_1^{-j})$. Let us write the integral in the second line of (88) as

$$\frac{1}{(2\pi i)^2}\sum_{\ell=0}^{k_0-1}\int\limits_{|x|=1+\varepsilon}\frac{g_\ell(x)}{x^{i+k_0-1}}\int\limits_{|y|=1+\varepsilon}\frac{q'_\ell(x,y)}{y^{j+k_0-1}Q(x,y)}dydx \tag{89}$$
$$+\frac{1}{(2\pi i)^2}\sum_{k=0}^{k_0-1}\int\limits_{|y|=1+\varepsilon}\frac{\widetilde{g}_k(y)}{y^{j+k_0-1}}\int\limits_{|x|=1+\varepsilon}\frac{q''_k(x,y)}{x^{i+k_0-1}Q(x,y)}dxdy$$
$$+\frac{1}{(2\pi i)^2}\int\limits_{|y|=1+\varepsilon}\frac{1}{y^{j+k_0-1}}\int\limits_{|x|=1+\varepsilon}\frac{f_{i_0,j_0}(x,y)}{x^{i+k_0-1}Q(x,y)}dxdy.$$

Using Lemma 19 (vii), we may rewrite the first term of (89) as

$$\frac{1}{(2\pi i)^2}\sum_{\ell=0}^{k_0-1}\int\limits_{|x|=1+\varepsilon}\frac{g_\ell(x)}{x^{i+k_0-1}}\int\limits_{|y|=Y_1(1+\varepsilon)-\eta}\frac{q'_\ell(x,y)}{y^{j+k_0-1}Q(x,y)}dydx, \tag{90}$$

for any $\eta>0$. Classically, the integral in (90) is bounded from above by (up to a multiplicative constant)

$$(1+\varepsilon)^i(Y_1(1+\varepsilon)-\eta)^j=o(x_1^{-i}+y_1^{-j}),$$

where the last equality is a consequence of Lemma 19 (viii), since $\eta>0$ may be taken as small as we want. We conclude similarly with the second and third terms of (89).

Step 6. Using (88) together with the computations just above, we deduce that

$$g\big((i_0,j_0)\to(i,j)\big)=\frac{\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)}{V_1}\pi_1(i)+\frac{\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)}{V_2}\pi_2(j)+o(x_1^{-i}+y_1^{-j}).$$

Finally, we use the fact that in Lemma 22, the constant c in the stationary measure asymptotics is non-zero, so we obtain the proof of Equation (10) of Theorem 2. ☐

Proof of Theorem 3. Let us rewrite (79) as

$$\pi_1(i) \sim A_1 x_1^{-i} \quad \text{and similarly} \quad \pi_2(j) \sim A_2 y_1^{-j}.$$

Plugging these estimates into (10) and using the definition (11) of t_0, one readily obtains

$$\frac{g\big((i_0,j_0)\to(i,j)\big)}{g\big((0,0)\to(i,j)\big)} \sim \frac{\big(\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)A_1/V_1\big)y_1^{j-it_0}+\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)A_2/V_2}{\big(\mathbb{P}_{(0,0)}\big(\mathcal{N}_1<\infty\big)A_1/V_1\big)y_1^{j-it_0}+\mathbb{P}_{(0,0)}\big(\mathcal{N}_2<\infty\big)A_2/V_2}. \tag{91}$$

Let us examine three different regimes when $i+j$ goes to infinity along an angular direction $j/i\to\tan\gamma$; see Figure 4. First, if $\gamma>\gamma_0$, then $j-it_0\to\infty$ and with (91), the limit Martin kernel equals

$$k(i_0,j_0) = \lim_{\substack{i+j\to\infty\\ j/i\to\tan\gamma}} \frac{g\big((i_0,j_0)\to(i,j)\big)}{g\big((0,0)\to(i,j)\big)} = \frac{\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)}{\mathbb{P}_{(0,0)}\big(\mathcal{N}_1<\infty\big)}.$$

For similar reasons, if now $\gamma<\gamma_0$, then $j-it_0\to-\infty$ and

$$k(i_0,j_0) = \frac{\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)}{\mathbb{P}_{(0,0)}\big(\mathcal{N}_2<\infty\big)}.$$

The limit case $\gamma=\gamma_0$ is the most interesting. The set of points $j-it_0$ in (91) lies on an additive subgroup of $\mathbb{R}$, namely, $\mathbb{Z}+t_0\mathbb{Z}$. If $t_0=\frac{n_0}{m_0}$ is rational, then the only possible limits for the Martin kernel are $(n\in\mathbb{Z})$

$$\frac{\big(\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)A_1/V_1\big)y_1^{n/m_0}+\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)A_2/V_2}{\big(\mathbb{P}_{(0,0)}\big(\mathcal{N}_1<\infty\big)A_1/V_1\big)y_1^{n/m_0}+\mathbb{P}_{(0,0)}\big(\mathcal{N}_2<\infty\big)A_2/V_2}.$$

In particular, the full Martin boundary is discrete. If $t_0\notin\mathbb{Q}$, then the subgroup $\mathbb{Z}+t_0\mathbb{Z}$ is dense, and any combination

$$\frac{\big(\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_1<\infty\big)A_1/V_1\big)u+\mathbb{P}_{(i_0,j_0)}\big(\mathcal{N}_2<\infty\big)A_2/V_2}{\big(\mathbb{P}_{(0,0)}\big(\mathcal{N}_1<\infty\big)A_1/V_1\big)u+\mathbb{P}_{(0,0)}\big(\mathcal{N}_2<\infty\big)A_2/V_2}$$

may appear in the limit, for any $u\in[0,\infty]$. The statement on the minimal Martin boundary is clear. ☐

Appendix A. Proof of Proposition 2

We give the proof of the first assertion of this proposition. The proof of the second assertion is quite similar. For $k\in\mathbb{N}$, consider the stopping time $T_1(k)=\inf\{n>0: X_1(n)=k\}$ as in (28), and

$$\tau_2(k)=\inf\{n>0: Y_2(n)=k\}. \tag{92}$$

In order to prove Proposition 2, we need three preliminary results, obtained in Lemmas 24, 25 and 26.

Lemma 24. *If $m_1 < 0$, then the quantity V_1 in (6) is well defined and for any $(k,\ell) \in \mathbb{N} \times \mathbb{Z}$, the random variable $Y_1(T_1(k))$ is integrable, with*

$$\mathbb{E}_{(k,\ell)}\big(Y_1(T_1(k))\big) = \ell + \frac{V_1}{\pi_1(k)}. \tag{93}$$

Proof. In order to prove that the quantity V_1 is well defined, it is sufficient to notice that by Assumption 2, there are two finite numbers $C > 0$ and $\delta > 0$, such that for any $(i,j) \in \mathbb{N} \times \mathbb{Z}$,

$$\mathbb{E}\big(|Y_1(1)|\big) \leqslant \frac{1}{\delta}\Big(\mathbb{E}\big(\exp(\delta Y_1(1))\big) + \mathbb{E}\big(\exp(-\delta Y_1(1))\big)\Big) \leqslant C. \tag{94}$$

The series in the definition of V_1 is therefore absolutely convergent, and consequently, V_1 is well defined.

To obtain (93), we introduce the following equivalent representation of the Markov process Z_1: let $\{\xi(i,j,n)\}_{i,j,n\in\mathbb{N}}$ be a family of random variables, which are mutually independent, independent on the Markov chain X_1, and such that for any $i,j,n \in \mathbb{N}$ and all $k \in \mathbb{Z}$,

$$\mathbb{P}\big(\xi(i,j,n) = k\big) = \frac{\mathbb{P}_{(i,0)}\big(X_1(1) = j, Y_1(1) = k\big)}{\mathbb{P}_{(i,0)}\big(X_1(1) = j\big)}.$$

Then, for a fixed $i \in \mathbb{N}$, given $(X_1(0), Y_1(0)) = (i,0)$, the distribution of the Markov chain $Z_1 = (X_1, Y_1)$ is identical to the distribution of the sequence $(X_1, \widetilde{Y}_1)$, where

$$\widetilde{Y}_1(0) = 0 \quad \text{and for } n \geqslant 1, \quad \widetilde{Y}_1(n) = \sum_{s=1}^{n} \xi\left(X_1(s-1), X_1(s), s\right).$$

Such a property holds because the transition probabilities of the Markov chain $Z_1 = (X_1, Y_1)$ are invariant with respect to vertical shifts.

Using such a representation of the Markov chain Z_1, we deduce that

$$\begin{aligned}
\mathbb{E}_{(k,\ell)}\big(Y_1(T_1(k))\big) - \ell &= \mathbb{E}_{(k,0)}\big(\widetilde{Y}_1(T_1(k))\big) \\
&= \mathbb{E}_{(k,0)}\left(\sum_{s=1}^{T_1(k)} \xi\left(X_1(s-1), X_1(s), s\right)\right) \\
&= \mathbb{E}_{(k,0)}\left(\sum_{s=1}^{T_1(k)} \sum_{i,j\in\mathbb{N}} \xi\left(i,j,s\right) \mathbb{1}_{\{X_1(s-1)=i, X_1(s)=j\}}\right).
\end{aligned}$$

Moreover, by independence of $\{\xi(i,j,n)\}_{i,j,n\in\mathbb{N}}$ and X_1, we conclude that

$$\mathbb{E}_{(k,\ell)}\big(Y_1(T_1(k))\big) - \ell = \mathbb{E}_{(k,0)}\left(\sum_{i,j\in\mathbb{N}} \mathbb{E}(\xi\left(i,j,1\right)) \sum_{s=1}^{T_1(k)} \mathbb{1}_{\{X_1(s-1)=i, X_1(s)=j\}}\right). \tag{95}$$

Following the definition of the random variables $\xi(i,j,1)$, one has

$$\big|\mathbb{E}\big(\xi(i,j,1)\big)\big| = \frac{\mathbb{E}_{(i,0)}\big(|Y_1(1)|\mathbb{1}_{\{X_1(1)=j\}}\big)}{\mathbb{P}_{(i,0)}\big(X_1(1) = j\big)}$$

as well as

$$\mathbb{E}_{(k,0)}\left(\sum_{s=1}^{T_1(k)} \mathbb{1}_{\{X_1(s-1)=i, X_1(s)=j\}}\right) = \frac{\pi_1(i)\mathbb{P}_{(i,0)}\big(X_1(1)=j\big)}{\pi_1(k)}.$$

By Fubini-Tonelli theorem, using (94) and going back to (95), it follows that

$$\begin{aligned}
\mathbb{E}_{(k,0)}&\left(\sum_{i,j\in\mathbb{N}} \big|\mathbb{E}\big(\xi(i,j,1)\big)\big| \sum_{s=1}^{T_1(k)} \mathbb{1}_{\{X_1(s-1)=i, X_1(s)=j\}}\right)\\
&\leqslant \sum_{i,j\in\mathbb{N}} \frac{\pi_1(i)\mathbb{E}_{(i,0)}\big(|Y_1(1)|\mathbb{1}_{\{X_1(1)=j\}}\big)}{\pi_1(k)}\\
&= \sum_{i\in\mathbb{N}} \frac{\pi_1(i)\mathbb{E}_{(i,0)}\big(|Y_1(1)|\big)}{\pi_1(k)}\\
&\leqslant \frac{C}{\pi_1(k)} < \infty.
\end{aligned}$$

Consequently, by Fubini theorem applied for the right-hand side of (95),

$$\begin{aligned}
\mathbb{E}_{(k,\ell)}\big(Y_1(T_1(k))\big) - \ell &= \sum_{i,j\in\mathbb{N}} \mathbb{E}\big(\xi(i,j,1)\big)\mathbb{E}_{(k,0)}\left(\sum_{s=1}^{T_1(k)} \mathbb{1}_{\{X_1(s-1)=i, X_1(s)=j\}}\right)\\
&= \sum_{i,j\in\mathbb{N}} \frac{\mathbb{E}_{(i,0)}\big(Y_1(1)\mathbb{1}_{\{X_1(1)=j\}}\big)\pi_1(i)}{\pi_1(k)} = \frac{V_1}{\pi_1(k)}.
\end{aligned}$$

□

Lemma 25. *If $m_1 < 0$, then for any $k \in \mathbb{N}$, the random variable $\sup_{0\leqslant s\leqslant T_1(k)} |Y_1(s)|$ is $\mathbb{P}_{(k,0)}$-integrable.*

Proof. Let us notice that for any $\varkappa > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}_{(k,0)}\left(\sup_{0\leqslant s\leqslant T_1(k)} |Y_1(s)| > \varkappa n\right) &\leqslant \mathbb{P}_{(k,0)}\big(T_1(k) > n\big)\\
&\quad + \mathbb{P}_{(k,0)}\left(T_1(k) \leqslant n, \sup_{0\leqslant s\leqslant T_1(k)} |Y_1(s)| > \varkappa n\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{P}_{(k,0)}\left(T_1(k) \leqslant n, \sup_{0\leqslant s\leqslant T_1(k)} |Y_1(s)| > \varkappa n\right) &\leqslant \mathbb{P}_{(k,0)}\left(\sup_{0\leqslant s\leqslant n} |Y_1(s)| > \varkappa n\right)\\
&\leqslant \sum_{s=1}^{n} \mathbb{P}_{(k,0)}\big(|Y_1(s)| > \varkappa n\big).
\end{aligned}$$

Using Markov inequality, for any $s \in \mathbb{N}$ and $\delta > 0$,

$$\begin{aligned}
\mathbb{P}_{(k,0)}\big(|Y_1(s)| > \varkappa n\big) &\leqslant \mathbb{P}_{(k,0)}\big(Y_1(s) > \varkappa n\big) + \mathbb{P}_{(k,0)}\big(-Y_1(s) > \varkappa n\big)\\
&\leqslant \exp(-\delta\varkappa n)\mathbb{E}_{(k,0)}\big(\exp(\delta Y_1(s)) + \exp(-\delta Y_1(s))\big).
\end{aligned}$$

Using the above inequality together with Assumption 2, it follows that there is a positive real number C such that for any $\kappa > 0$, $s \in \{1, \dots, n\}$ and $\delta > 0$ small enough,

$$\mathbb{P}_{(k,0)}\big(|Y_1(s)| > \varkappa n\big) \leqslant 2\exp(-\delta\varkappa n)C^s \leqslant 2\exp\big(-\delta\varkappa n + \log(C)n\big).$$

Hence, setting $\varkappa = 2\frac{\log C}{\delta}$, we have

$$\mathbb{P}_{(k,0)}\left(\sup_{0\leqslant s\leqslant T_1(k)} |Y_1(s)| > \varkappa n\right) \leqslant \mathbb{P}_{(k,0)}\big(T_1(k) > n\big) + n\exp(-\delta\varkappa n/2). \tag{96}$$

Since, classically,

$$\sum_{n=0}^{\infty} \mathbb{P}_{(k,0)}\big(T_1(k) > n\big) = \mathbb{E}_{(k,0)}\big(T_1(k)\big) < \infty,$$

it follows from (96) that the series

$$\sum_{n=0}^{\infty} \mathbb{P}_{(k,0)}\left(\sup_{0\leqslant s\leqslant T_1(k)} |Y_1(s)| > \varkappa n\right)$$

converges, and consequently, the random variable $\sup_{0\leqslant s\leqslant T_1(k)} |Y_1(s)|$ is integrable. □

Consider an increasing sequence of stopping times $\{t_n\}$ defined by

$$t_0 = 0, \quad t_1 = T_1(k) \quad \text{and} \quad t_{n+1} = \inf\{s > t_n : X_1(s) = k\}.$$

Lemma 26. *If $m_1 < 0$, then for any $(k,\ell) \in \mathbb{N}\times\mathbb{Z}$, $\mathbb{P}_{(k,\ell)}$-a.s.,*

$$\lim_{n\to\infty} \frac{t_{n+1} - t_n}{n} = 0 \tag{97}$$

and

$$\lim_{n\to\infty} \sup_{t_n\leqslant s<t_{n+1}} \frac{|Y_1(s) - Y_1(t_n)|}{n} = 0. \tag{98}$$

Proof. Remark that according to the definition of the sequence $\{t_n\}$, by the strong Markov property, for any $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\mathbb{P}_{(k,\ell)}\left(\frac{t_{n+1} - t_n}{n} > \varepsilon\right) = \mathbb{P}_{(k,\ell)}\left(\frac{\tau_1(k)}{\varepsilon} > n\right).$$

Since for $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{(k,\ell)}\left(\frac{\tau_1(k)}{\varepsilon} > n\right) \leqslant \frac{\mathbb{E}_{(k,\ell)}(\tau_1(k))}{\varepsilon} < \infty,$$

this proves that for any $\varepsilon > 0$, the series

$$\sum_{n=1}^{\infty} \mathbb{P}_{(k,\ell)}\left(\frac{t_{n+1} - t_n}{n} > \varepsilon\right)$$

converges, and consequently, (97) holds.

The proof of (98) is similar: for any $n \in \mathbb{N}$ and $\varepsilon > 0$, according to the definition of $\{t_n\}$ and by the strong Markov property,

$$\mathbb{P}_{(k,\ell)}\left(\sup_{t_n\leqslant s<t_{n+1}} \frac{|Y_1(s) - Y_1(t_n)|}{n} > \varepsilon\right) = \mathbb{P}_{(k,\ell)}\left(\sup_{0\leqslant s<T_1(k)} \frac{|Y_1(s) - \ell|}{\varepsilon} > n\right).$$

Since for $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{(k,\ell)} \left(\sup_{0 \leqslant s < T_1(k)} \frac{|Y_1(s) - \ell|}{\varepsilon} > n \right) \leqslant \frac{\mathbb{E}_{(k,\ell)} \left(\sup_{0 \leqslant s < T_1(k)} |Y_1(s) - \ell| \right)}{\varepsilon}$$

and by Lemma 25, the right-hand side of the last inequality is finite, we conclude that for any $\varepsilon > 0$, the series

$$\sum_{n=1}^{\infty} \mathbb{P}_{(k,\ell)} \left(\sup_{t_n \leqslant s < t_{n+1}} \frac{|Y_1(s) - Y_1(t_n)|}{n} > \varepsilon \right)$$

converges, and consequently, (98) holds. □

We are ready to complete the proof of Proposition 2. Suppose that $X_1(0) = k \in \mathbb{N}$ and $Y_1(0) = \ell \in \mathbb{Z}$, and let us prove that $\mathbb{P}_{(k,\ell)}$-a.s.,

$$\lim_{s \to \infty} \frac{Y_1(s)}{s} = V_1. \tag{99}$$

By the strong Markov property, the random variables $\{\eta_n\}_{n \geqslant 0} = \{t_{n+1} - t_n\}_{n \geqslant 0}$ are identically distributed and mutually independent. Moreover, the Markov chain X_1 being positive recurrent,

$$\mathbb{E}(\eta_n) = \frac{1}{\pi_1(k)} < \infty,$$

and consequently, by the strong law of large numbers, $\mathbb{P}_{(k,\ell)}$-a.s.,

$$\lim_{n \to \infty} \frac{t_n}{n} = \frac{1}{\pi_1(k)}.$$

Remark furthermore that the sequence $\{S_n\} = \{Y_1(t_n)\}$ is a homogeneous random walk on $\mathbb{Z}$, and for any $n \in \mathbb{N}$, by (93),

$$\mathbb{E}_{(k,\ell)} \big(Y_1(t_{n+1}) - Y_1(t_n) \big) = \mathbb{E}_{(k,0)} \big(Y_1(t_1) \big) = \frac{V_1}{\pi_1(k)}.$$

Hence, using again the strong law of large numbers, we conclude that $\mathbb{P}_{(k,\ell)}$-a.s.,

$$\lim_{n \to \infty} \frac{Y_1(t_n)}{t_n} = \lim_{n \to \infty} \frac{Y_1(t_n)}{n} \times \frac{n}{t_n} = \frac{V_1}{\pi_1(k)} \times \pi_1(k) = V_1. \tag{100}$$

Now, to get (99), for a given $s \in \mathbb{N}$ we consider $n(s) \in \mathbb{N}$ such that

$$t_{n(s)} \leqslant s < t_{n(s)+1},$$

and we notice that

$$\begin{aligned} \left| \frac{Y_1(t_{n(s)})}{t_{n(s)}} - \frac{Y_1(s)}{s} \right| &\leqslant \left| \frac{Y_1(t_{n(s)})}{t_{n(s)}} \right| \times \frac{s - t_{n(s)}}{s} + \left| \frac{Y_1(t_{n(s)}) - Y_1(s)}{s} \right| \\ &\leqslant \left| \frac{Y_1(t_{n(s)})}{t_{n(s)}} \right| \times \frac{t_{n(s)+1} - t_{n(s)}}{t_{n(s)}} + \left| \frac{Y_1(t_{n(s)}) - Y_1(s)}{t_{n(s)}} \right|. \end{aligned}$$

Moreover, according to the definition of the sequence $\{t_n\}$, for any $n \in \mathbb{N}$, $t_{n+1} - t_n \geqslant 1$ and $t_n < \infty$ because the Markov chain X_1 is recurrent. From this it follows that

$$t_n \geqslant n, \quad \forall n \in \mathbb{N},$$

and $n(s) \to \infty$ when $s \to \infty$. Hence,

$$\left|\frac{Y_1(t_{n(s)})}{t_{n(s)}} - \frac{Y_1(s)}{s}\right| \leqslant \left|\frac{Y_1(t_{n(s)})}{t_{n(s)}}\right| \times \frac{t_{n(s)+1} - t_{n(s)}}{n(s)} + \left|\frac{Y_1(t_{n(s)}) - Y_1(s)}{n(s)}\right|,$$

where by (100), $\mathbb{P}_{(k,\ell)}$-a.s.

$$\lim_{s\to\infty} \frac{Y_1(t_{n(s)})}{t_{n(s)}} = V_1,$$

and by Lemma 26, $\mathbb{P}_{(k,\ell)}$-a.s.

$$\lim_{s\to\infty} \frac{t_{n(s)+1} - t_{n(s)}}{n(s)} = 0$$

and

$$\lim_{s\to\infty} \frac{|Y_1(t_{n(s)}) - Y_1(s)|}{n(s)} = 0.$$

This proves that $\mathbb{P}_{(k,\ell)}$-a.s.

$$\lim_{s\to\infty} \left(\frac{Y_1(t_{n(s)})}{t_{n(s)}} - \frac{Y_1(s)}{s}\right) = 0$$

and consequently, using again (100), we obtain (99).

Appendix B. Glossary of the hitting times

Throughout the paper, we introduced and made use of the following first hitting times:

$$\begin{cases} \tau(k,\ell) &= \inf\{n > 0 : Z(n) = (k,\ell)\}, & \text{(see (16))}, \\ \tau &= \inf\{n > 0 : Y(n) < k_0\}, & \text{(see (17))}, \\ \tau_1(k,\ell) &= \inf\{n > 0 : Z_1(n) = (k,\ell)\}, & \text{(see (18))}, \\ \tau_1 &= \inf\{n > 0 : Y_1(n) < k_0\}, & \text{(see (19))}, \\ T_1(k) &= \inf\{n > 0 : X_1(n) = k\}, & \text{(see (28))}, \\ T_1^k &= \inf\{n > 0 : X_1(n) \leqslant (k_0 - 1) \vee k\}, & \text{(see (29))}, \\ T^k &= \inf\{n > 0 : X(n) \leqslant (k_0 - 1) \vee k\}, & \text{(see (46))}, \\ \tau_2(k) &= \inf\{n > 0 : Y_2(n) = k\}, & \text{(see (92))}. \end{cases}$$

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