LATTICE WALKS CONFINED TO AN OCTANT IN DIMENSION 3: (NON-)RATIONALITY OF THE SECOND CRITICAL EXPONENT

LUC HILLAIRET, HELEN JENNE, AND KILIAN RASCHEL

ABSTRACT. In the field of enumeration of walks in cones, it is known how to compute asymptotically the number of excursions (finite paths in the cone with fixed length, starting and ending points, using jumps from a given step set). As it turns out, the associated critical exponent is related to the eigenvalues of a certain Dirichlet problem on a spherical domain. An important underlying question is to decide whether this asymptotic exponent is a (non-)rational number, as this has important consequences on the algebraic nature of the associated generating function. In this paper, we ask whether such an excursion sequence might admit an asymptotic expansion with a first rational exponent and a second non-rational exponent. While the current state of the art does not give any access to such many-term expansions, we look at the associated continuous problem, involving Brownian motion in cones. Our main result is to prove that in dimension three, there exists a cone such that the heat kernel (the continuous analogue of the excursion sequence) has the desired rational/non-rational asymptotic property. Our techniques come from spectral theory and perturbation theory. More specifically, our main tool is a new Hadamard formula, which has an independent interest and allows us to compute the derivative of eigenvalues of spherical triangles along infinitesimal variations of the angles.

1. INTRODUCTION

The model and our main question. A lattice walk is a sequence of points $P_0, P_1, \ldots, P_n$ of $\mathbb{Z}^d$, $d \geqslant 1$. The points P_0 and P_n are its starting and end points, respectively, the consecutive differences $P_{i+1} - P_i$ its steps, and n is its length. Given a set $\mathcal{S} \subset \mathbb{Z}^d$, called the step set, a set $C \subset \mathbb{Z}^d$ called the domain (which in this paper will systematically be a cone), and elements P and Q of C, we are interested in the number $e(P, Q; n)$ of walks (or excursions) of length n that start at P, have all their steps in $\mathcal{S}$, have all their points in C, and end at Q. In the present note, the main problem we would like to address is the following: does there exist a walk model (i.e., a step set and a cone in $\mathbb{R}^d$) such that as $n \to \infty$, one has the asymptotics

$$(1) \qquad e(P, Q; n) = \rho^n \cdot \left(K_1 \cdot n^{\alpha_1} + K_2 \cdot n^{\alpha_2} + \cdots + K_p \cdot n^{\alpha_p} + o(n^{\alpha_p}) \right),$$

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with some exponential growth $\rho > 0$ and critical exponents such that $\alpha_1, \ldots, \alpha_{p-1} \in \mathbb{Q}$ and $\alpha_p \notin \mathbb{Q}$? (The constants $K_1, \ldots, K_p$ are assumed to be non-zero.) We now present the context and explain our motivations to look at this particular problem.

Asymptotics of the excursion sequence and relation to D-finiteness. Is there a simple formula for $e(P, Q; n)$ in terms of the coordinates of P, Q and the length n of the walk? If not, can we at least say something about the asymptotic behaviour (1) of these numbers as n goes to infinity? A first step towards answering these questions can be done by considering the excursion generating function

$$(2) \qquad e_{P,Q}(t) = \sum_{n \geqslant 0} e(P, Q; n) t^n \in \mathbb{Q}[[t]]$$

that is associated with these numbers and determining whether it is algebraic, or, if not, whether it is at least D-finite. Recall that a series is D-finite if it satisfies a non-trivial linear differential equation with polynomial coefficients. Knowing that a given series is D-finite not only implies nice computational properties of its coefficients, but also allows us to classify the combinatorial model according to the complexity of the underlying generating function. There has recently been a dense literature around the above questions, in relation with the probabilistic model of random walks in cones.

As it turns out, there is a strong relation between D-finiteness of a given series and the asymptotic behavior of its coefficients. For example, the following statement (recalled in [7, Thm 3]) is a consequence of results by André, Chudnovski and Katz:

Lemma 1. *Let $(e(n))_{n \geqslant 0}$ be an integer-valued sequence whose n-th term behaves asymptotically like*

$$(3) \qquad e(n) \sim K \cdot \rho^n \cdot n^\alpha$$

for some real positive constants K and ρ. If the singular exponent α is irrational, then the generating function $e(t) = \sum_{n=0}^{\infty} e(n) t^n$ is not D-finite.

Given the above result, it is natural to ask whether one may compute and study the rationality of the critical exponent α in the asymptotics (3) of the excursion sequence $e(P, Q; n)$ (equivalently, the dominant term in the asymptotics (1)).

In dimension 1, the combinatorial model of walks in cones reduces to that of walks confined to the positive half-line, as studied e.g. in [2]. In this context, it is well known that only simple exponents appear in the dominant asymptotics, namely $\alpha = 0$, $-\frac{1}{2}$ or $-\frac{3}{2}$ (depending on the drift of the model), and their translations by integers in the complete asymptotic expansion (1). Accordingly, there is nothing to say from the perspective of the rationality of asymptotic exponents. We remark that these simple exponents are deduced from the algebraicity of the associated generating function (2) and classical transfer theorems (singularity analysis).

Given a cone in higher dimension $d \geqslant 2$, the generating function (2) is in general not algebraic (and even non-D-finite, see [20, 7] for the case of the quarter plane in dimension 2), and the first problem is to access the critical exponent α. This result is obtained by Denisov and Wachtel [11]: for a large class of cones in arbitrary dimension, they derive the

one-term asymptotics (3) for the excursion sequence $e(P, Q; n)$. In particular, they show that [11, Eq. (12)]

$$\alpha = -\sqrt{\lambda_1 + (d/2 - 1)^2} - 1, \tag{4}$$

where d is the dimension and λ_1 is interpreted as the principal Dirichlet eigenvalue for the Laplace-Beltrami operator on the subdomain of the sphere $\mathbb{S}^{d-1}$ given by

$$(LC) \cap \mathbb{S}^{d-1}, \tag{5}$$

with C being the domain of confinement (typically an orthant $\mathbb{R}_+^d$) and L a linear application, which depends on the model. One should not be surprised by the presence of the linear transform L in (5): as a matter of comparison, the classical central limit theorem for random walks in $\mathbb{R}^d$ involves the drift and the covariance matrix of the process, so as to put the random walk in the domain of attraction of a standard Brownian motion (here, standard means without drift and with identity covariance matrix). Similarly, the application L above appears so as to take into account the drift and the covariance matrix of the combinatorial model under consideration.

Some key ingredients in Denisov and Wachtel's proof are a coupling of random walk by Brownian motion and then a use of older results in the probabilistic literature on exit times for Brownian motion [10, 3, 24] (the eigenvalue λ_1 already appearing in the study of Brownian motion in cones).

Accordingly, all the complexity of the excursion (one-term) asymptotics (3) is contained in the principal eigenvalue λ_1.

Dimension 2. Regarding the combinatorial model of walks in the quarter plane, the domain (5) simply becomes an arc of circle, see Figure 1 for a few examples. More precisely, if the walk is driftless and has identity covariance matrix, then L is just the identity and (5) is a quarter of circle. For other walk models, using the expression of the linear transform L, the arc has opening $\beta \in (0, \pi)$, which one may express as $\arccos(-r)$, where r is an algebraic number which is easily computed from the model; see [7] for more details.

As it turns out, the principal eigenvalue (and in fact the whole spectrum) of arcs of circles is known. More precisely, if the cone has opening β, then $\lambda_1 = (\frac{\pi}{\beta})^2$, and more generally the j-th eigenvalue is given by $\lambda_j = (j\frac{\pi}{\beta})^2$. Consequently, using (4), one deduces that the asymptotic exponent α is known and is equal to

$$\alpha = -\frac{\pi}{\beta} - 1 = -\frac{\pi}{\arccos(-r)} - 1.$$

For instance, for the model on the left on Figure 1 (called a scarecrow in [7]), one has $r = 1/4$ and thus $\alpha = -\frac{\pi}{\arccos(-1/4)} - 1$, which can be proved to be non-rational [7].

Following this approach, the authors of [7] obtain that for a list of 51 (unweighted, having infinite group and small steps) models, α is non-rational, and so, using Lemma 1, these 51 models admit non-D-finite generating functions. In the context of unweighted quadrant lattice walks, it is remarkable that the converse statement is also true: in other words, the generating function (2) of the 74 non-singular, unweighted quadrant lattice walks is D-finite if and only if the principal exponent is rational.

FIGURE 1. Three examples of walks in dimension 2 confined to the quarter plane $C = \mathbb{R}_+^2$, with the associated domain $(LC) \cap \mathbb{S}^1$ as in (5). Their critical exponent α in (3) is as follows: on the left, $\alpha = -3$; for the second model $\alpha = -\frac{\pi}{\arccos(-1/4)} - 1 \notin \mathbb{Q}$, see [7]; on the right $\alpha = -5$, see [6].

This equivalence (between D-finiteness of the generating function and rationality of the critical exponent) is a priori not true in general: the authors of [6] construct several models (one of them is represented on Figure 1, right) for which α is rational but the generating function is conjectured to be non-D-finite. See Table 2 in [6] for more examples.

With this in mind, our question in dimension 2 would be to see whether there exist quadrant walk models such that the associated excursion sequence admits the asymptotics (1), with $\alpha_1, \ldots, \alpha_{p-1} \in \mathbb{Q}$ and $\alpha_p \notin \mathbb{Q}$. Such a statement would also lead to non-D-finiteness results, by a generalization of Lemma 1 to many-term asymptotic expansions. See in particular the works [13, 14], where this generalization is mentioned.

As we will explain later, we conjecture that the above rationality/non-rationality phenomenon does not occur in dimension 2.

Dimension 3. We now explore the case of dimension 3. First, the domain (5) to consider is the trace on the sphere $\mathbb{S}^2$ of $L\mathbb{R}_+^3$, which by construction is a spherical triangle, see Figure 2 for a few examples. In other words, in dimension 3, one has to understand the principal eigenvalue λ_1 of spherical triangles. This connection between three-dimensional positive lattice walks and spherical triangles has been studied in [5], see also [21] in relation with a Brownian pursuit problem.

While in dimension 2, it was possible to compute the whole spectrum for the Laplace-Beltrami problem with Dirichlet conditions on the domain (5), and in addition we had nice formulas for all eigenvalues and eigenfunctions (recall that $\lambda_j = (j\frac{\pi}{\beta})^2$ in the planar case), this is no longer the case in dimension 3. More precisely, given a generic spherical triangle, it is in general impossible to compute in closed form any of its eigenvalues. To summarize, up to our knowledge, there are only two kinds of exceptional spherical triangles which admit eigenvalues in closed form:

- Spherical triangles corresponding to tilings of the sphere [4]. Notice that tilings do not all lead to an explicit spectrum: for instance, the one on the right on Figure 2 (called the tetrahedral tiling) cannot be solved in an explicit manner, as it does

FIGURE 2. Two tilings of the sphere by triangles. The left one corresponds to the simple walk, with jumps $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. The tiling on the right is associated to the following model, known as 3D Kreweras model: $\{(-1, 0, 0), (0, -1, 0), (0, 0, -1), (1, 1, 1)\}$. The rationality of its critical exponent α is still unknown.

not admit the right parity). Specifically, the angles of these triangles should take one of the following values: $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5})$ or $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{r})$, with some integer $n \geqslant 2$.
- Arbitrary birectangular triangles, i.e., triangles admitting two right angles $\frac{\pi}{2}$ and one arbitrary angle $\beta \in (0, 2\pi)$; see [25, 26, 23].

In this three-dimensional context, our question takes the following form: does there exist an octant walk model such that the associated excursion sequence admits the asymptotics (1) with $\alpha_1, \ldots, \alpha_{p-1} \in \mathbb{Q}$ and $\alpha_p \notin \mathbb{Q}$?

The heat kernel of cones. To answer our main question, one intrinsic difficulty is to know a many-term asymptotic expansion of the form of (1). And indeed, such asymptotics are not available in the literature in general (except in a few very particular cases, which are the simplest cases, so precisely those with a complete asymptotic expansions with rational exponents, see e.g. [8]).

As a consequence, in order to progress on our question, we will reason by analogy between the discrete setting (random walk) and the continuous setting (Brownian motion), and we will solve the analogous question in the Brownian framework.

First of all, the quantity analogous to the number of excursions $e(P, Q; n)$ is called the (continuous) heat kernel of the cone, which, as we shall see, admits an expression in closed-form (7) and explicit complete asymptotic expansions.

The heat kernel $p^C(x, y; t)$ of a cone (and actually of any domain) C admits the following probabilistic interpretation: it is the probability density function of the transition probability kernel

$$(6) \qquad p^C(x, y; t)dy = \mathbb{P}_x(B_t \in dy, \tau_C > t),$$

where the Brownian motion is denoted by B_t and τ_C is the first exit time from the cone C, that is, $\tau_C = \inf\{t > 0 : B_t \notin C\}$. Letting $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ denote the eigenvalues of the Laplace-Beltrami operator with Dirichlet conditions on the domain $C \cap \mathbb{S}^{d-1}$, its

explicit expression is given by [3]

$$(7) \qquad p^C(x,y;t) = \frac{\exp\left(-\frac{|x|^2+|y|^2}{2t}\right)}{t(|x|\cdot|y|)^{d/2-1}} \sum_{j=1}^{\infty} I_{\alpha_j}\left(\frac{|x|\cdot|y|}{t}\right) m_j\left(\frac{x}{|x|}\right) m_j\left(\frac{y}{|y|}\right),$$

where $\alpha_j = \sqrt{\lambda_j + (d/2-1)^2}$, m_j is the associated normalized eigenfunction, and I_ν is the modified Bessel function of order ν, which admits the expression

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)^{\nu+2m}.$$

The following result may be found in [9, Thm 2.3]:

Lemma 2. *For any dimension $d \geqslant 1$ and any cone regular enough, the heat kernel $p^C(x,y;t)$ in (6) admits a complete asymptotic expansion of the form*

$$(8) \qquad p^C(x,y;t) = K_1 \cdot t^{-\alpha_1} + K_2 \cdot t^{-\alpha_2} + \cdots + K_p \cdot t^{-\alpha_p} + o(t^{-\alpha_p}),$$

where

- *the order p of the expansion is arbitrary large;*
- *the constants K_i depend on x and y, i.e., $K_i = K_i(x,y)$;*
- *the exponents α_i are independent of x and y, and $\alpha_1 < \alpha_2 < \cdots < \alpha_p$;*
- *$\alpha_i = \sqrt{\lambda_j + (d/2-1)^2} + k$, where λ_j is an eigenvalue and k is a positive integer.*

In this new (and last!) setting, our question may be formulated as follows: is it possible in the asymptotics (8) to have first rational exponents $\alpha_1, \ldots, \alpha_{p-1}$ and a non-rational α_p?

Statements of our main results. As the following proposition establishes, our question is easily solved in dimension 2, and the answer happens to be negative.

Proposition 3. *In dimension $d = 2$, the exponents α_i appearing in the asymptotics (8) of the heat kernel are simultaneously all rational or non-rational.*

Proof. Using that $d = 2$ and $\lambda_j = (j\frac{\pi}{\beta})^2$, it follows from Lemma 2 that the exponents α_i may be expressed as $j\frac{\pi}{\beta} + k$, where j and k are positive integers. Clearly, when j and k vary, these numbers are either all rational or all non-rational. $\square$

Accordingly, we also conjecture that we cannot construct any discrete model having this rationality/non-rationality property (with sufficiently many moment conditions).

Although we shall not elaborate on this here, we would like to mention that, based on the above two-dimensional result, it should be easy to give an example to our rationality/non-rationality phenomenon in dimension 4, seeing $\mathbb{R}^4$ as a product of two planes and defining on each plane a different model, one with $\frac{\pi}{\beta_1} \in \mathbb{Q}$ and the second one with $\frac{\pi}{\beta_2} \in \mathbb{Q}$. We thank Andrew Elvey Price for this suggestion.

So we have to move to dimension 3. Our main theorem in this paper is the following:

Theorem 1. *There exists a 3D cone such that the heat kernel admits the asymptotics (8), with first rational exponents $\alpha_1, \ldots, \alpha_{p-1}$ and then a non-rational exponent α_p.*

Theorem 1 is a rather direct consequence of the following result:

Theorem 2. *There exists $t_0 > 0$ and a real analytic function b defined on $(-t_0, t_0)$, such that the one parameter family of triangles $(T_t)_{t \in (-t_0, t_0)}$ that have one side of length $\frac{\pi}{2}$ and adjacent angles with values*

$$\frac{\pi}{2} + t, \frac{\pi}{2} + b(t),$$

satisfies, for the Dirichlet Laplace operator,

- *the first eigenvalue $\lambda_1(t)$ of T_t is constant: $\forall t \in (-t_0, t_0), \ \lambda_1(t) = 12$,*
- *the second eigenvalue $\lambda_2(t)$ admits the first order approximation:*

$$\lambda_2(t) = 30 - 22\sqrt{3} \cdot |t| + o(t).$$

2. The spectrum of spherical triangles

We prove Theorem 2 by studying the first eigenvalues as functions on the set $\mathcal{M}$ of spherical triangles with one side of length $\frac{\pi}{2}$. We first show that the level sets of the first eigenvalue λ_1 are analytic curves in $\mathcal{M}$. Denote by T_* the equirectangle triangle, see Figure 3 (left). Restricting to the curve on which the first eigenvalue is constant and equal to $12 = \lambda_1(T_*)$, we compute the derivatives of the second and third eigenvalue branches at T_*. Since the latter derivatives do not vanish, the theorem will be proved.

This strategy of proof relies heavily on analytic perturbation theory (see [17]) and similar techniques which have been used by the authors of [23] to study the spectral gap of spherical triangles. The reader new to analytic perturbation theory may also find [12] as a useful reference giving a similar application of this theory.

2.1. **The set of spherical triangles and the associated spectral problem.** Let A_* and B_* be two points at distance $\frac{\pi}{2}$ on the unit sphere in $\mathbb{R}^3$. We choose one of the two hemispheres that have A_* and B_* on its boundary and denote by $\mathcal{M}$ the set of triangles whose vertices are A_*, B_* and C, where C is any point of that hemisphere. For any T in $\mathcal{M}$, we denote by a the length of the side opposite to A_* (resp. b and c) any by α the angle at A_* (resp. β at B_* and γ at C). Figure 3 summarizes these notations.

Remark 1. *Strictly speaking, to properly define the set of triangles with one side of length $\frac{\pi}{2}$ we should mod out by the involution $(\alpha, \beta) \leftrightarrow (\beta, \alpha)$. We do not need this subtlety here and may freely work on $\mathcal{M}$.*

The set $\mathcal{M}$ is naturally parametrized by $(\alpha, \beta) \in (0, \pi) \times (0, \pi)$ and we will denote by $T(\alpha, \beta)$ the corresponding triangle. Analyticity on $\mathcal{M}$ means analyticity in (α, β).

We also define the distance between two triangles T and T' by

$$d(T, T') = \max(|\alpha - \alpha'|, |\beta - \beta'|).$$

We let $T_* = T(\frac{\pi}{2}, \frac{\pi}{2})$ and A_*, B_*, C_* its vertices.

For any fixed β, when α goes to 0, the triangle $T(\alpha, \beta)$ degenerates onto the arc A_*B_*, and when α goes to π, it degenerates onto $\mathcal{D}_\beta$: the digon (or spherical lune) of opening angle β, see Figure 4 (left).

FIGURE 3. On the left: the equirectangle triangle $T_* = T(\frac{\pi}{2}, \frac{\pi}{2})$. On the right: a generic triangle $T(\alpha, \beta)$ in $\mathcal{M}$.

FIGURE 4. On the left: a digon (or spherical lune) is a domain bounded by two great circles. On the right: the spherical coordinates (r, θ)

We will use (spherical) polar coordinates at A_*: the point $M(r, \theta)$ is at distance r along the geodesic that emanates from A_*, making the angle θ with the arc A_*B_*; see Figure 4 (right).

The side $[B_*, C]$ is parametrized, in these polar coordinates, by the mapping $\theta \mapsto L_\beta(\theta)$ that is implicitly defined by the following application of the cotangent four-part formula: (that is simplified using that the distance between A_* and B_* is $\frac{\pi}{2}$)

$$0 = \cot L_\beta(\theta) - \cot \beta \sin \theta.$$

This equation can be solved by setting

$$\forall \beta, \theta \in (0, \pi), \quad L_\beta(\theta) = \operatorname{arccot}(\cot \beta \sin \theta),$$

with arccot the reciprocal function to cot with values in $(0, \pi)$. The mapping $(\beta, \theta) \mapsto L_\beta(\theta)$ is thus analytic on $(0, \pi)^2$ and, for any $\beta \in (0, \pi)$, the mapping $\theta \mapsto L_\beta(\theta)$ extends smoothly to $\mathbb{R}$.

Thus we have the parametrization:

$$(9) \qquad T(\alpha, \beta) = \{(r, \theta) : \quad \theta \in (0, \alpha), \ r \in (0, L_\beta(\theta))\}.$$

In these polar coordinates, the spherical metric reads $g = dr^2 + \sin^2(r)d\theta^2$, the area element is $\sin r \, dr d\theta$ and the Dirichlet energy quadratic form for the triangle $T = T(\alpha, \beta)$ is, for any $u \in C_0^\infty(T)$,

$$(10) \qquad q(u) = \int_T \left[|\partial_r u(r, \theta)|^2 + \frac{1}{\sin^2 r} |\partial_\theta u(r, \theta)|^2 \right] \sin r \, dr d\theta.$$

We also denote by n the Riemannian L^2 norm on T:

$$(11) \qquad n(u) = \int_T |u(r, \theta)|^2 \sin r \, dr d\theta.$$

We will abuse notation by also using q and n to denote the bilinear forms that are canonically associated with q and n.

We now explain how to associate a self-adjoint operator (that we call the Dirichlet Laplace operator) to this setting. The procedure is quite standard and we refer the reader to [22] for more details. It is well-known that when Q is a bounded quadratic form on a Hilbert space $\mathcal{H}$ with scalar product n, there exists a unique associated self-adjoint operator that satisfies

$$\forall x \in \mathcal{H}, Q(x) = n(Ax, x).$$

The latter statement can be extended to closed unbounded quadratic forms. However, with the definitions above and since the quadratic form q is defined on $C_0^\infty(T)$ only, it is not closed. In order to prove that the quadratic form q is closable, we remark that, using integration by parts, there exists a partial differential operator P such that

$$\forall u \in C_0^\infty(T), \quad q(u) = n(Pu, u).$$

Moreover, the operator P, with domain $C_0^\infty(T)$ is formally symmetric so that we can use the Friedrichs extension procedure. As a result, the Dirichlet Laplace operator is obtained as follows. We first define $H_0^1(T)$ to be the completion of $C_0^\infty(T)$ with respect to the quadratic form $q + n$. The quadratic form q with domain H_0^1 is now closed and the unique associated self-adjoint operator is the Dirichlet Laplace operator on T. We denote it by Δ (observe that, by construction, Δ is a non-negative operator). Despite the corners, the injection from $H_0^1(T)$ into $L^2(T)$ is still compact so that the spectrum of Δ consists solely of eigenvalues of finite multiplicity. The construction implies that a function u is an eigenfunction of Δ with eigenvalue λ if and only if the following system is satisfied:

$$(12) \qquad \begin{cases} u \in H_0^1(T), \\ \forall v \in H_0^1(T), \quad q(u, v) = \lambda n(u, v). \end{cases}$$

Remark 2. *Elaborating on the results of appendix A, it can be proved that the eigenfunctions of the latter eigenvalue problem do vanish on the sides of the triangles, hence justifying the "Dirichlet" appellation.*

2.2. **Analyticity of the spectrum.** For each triangle in $\mathcal{M}$, the spectral problem (12) gives a spectrum that is usually organized in a non-decreasing sequence:

$$\lambda_1(T) < \lambda_2(T) \leqslant \cdots \leqslant \lambda_n(T) \leqslant \cdots$$

Each eigenvalue is repeated according to its multiplicity, and we have used the known fact that the first eigenvalue $\lambda_1(T)$ is simple.

The theory of analytic perturbations gives conditions under which the spectrum of a family of such spectral problems depends analytically on its parameters. We refer to [17] for a complete account on the theory and we now wish to apply the theory when the parameters (α, β) vary.

Let $T_0 = T(\alpha_0, \beta_0)$ be a triangle in $\mathcal{M}$, and let $T = T(\alpha, \beta)$ be another triangle in a small neighbourhood of T_0. We recall that L_{β_0} and L_β are the functions that are used to describe T_0 and T in polar coordinates, see (9).

Analytic perturbation theory applies to a family of quadratic forms on a fixed Hilbert space. It cannot be used directly here since the spectral problems associated with T and T_0 are not defined in the same Hilbert space, and the corresponding quadratic forms do not have the same domain. In order to circumvent this problem, we first define a diffeomorphism between T_0 and T. We want this diffeomorphism to depend analytically on (α, β), but it is actually not necessary to define very precisely what the latter means: analyticity will be checked on the expression of the quadratic forms in the end.

In order to get Hadamard variational formulas (which we will obtain in Theorems 3 and 4), it is convenient to choose our diffeomorphisms as follows. We choose χ to be a smooth non-negative and non-increasing function on $\mathbb{R}$ such that χ is identically 1 on $(-\infty, \frac{1}{3})$ and identically 0 on $(\frac{2}{3}, \infty)$, and we fix some $\varepsilon > 0$. Let Φ be the mapping defined on T_0 by

$$\Phi(r, \theta) = (R, \Theta),$$

with

$$\begin{cases} \Theta(r, \theta) &= \theta + (\alpha - \alpha_0)\chi\left(\frac{\alpha_0 - \theta}{\varepsilon}\right), \\ R(r, \theta) &= r + (L_\beta \circ \Theta(r, \theta) - L_{\beta_0}(\theta))\chi\left(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\right). \end{cases}$$

This mapping actually depends on α, β, and ε, i.e., $\Phi = \Phi_{\alpha,\beta}^{(\varepsilon)}$, but for readability, the notation does not reflect it. We also set $\ell_\beta = L_\beta \circ \Theta$ and $\ell_0 = L_{\beta_0}$ and observe that these functions depend only on θ.

We now pull back the spherical metric on T to T_0 using this diffeomorphism. We thus introduce the Jacobian matrix of Φ:

$$\text{Jac}\,\Phi_{|(r,\theta)} = \begin{pmatrix} A(r, \theta) & C(r, \theta) \\ 0 & B(r, \theta) \end{pmatrix},$$

where we have set:

$$\begin{cases} A(r, \theta) &= \partial_r R(r, \theta) &= 1 - (\ell_\beta(\theta) - \ell_0(\theta))\frac{1}{\varepsilon}\chi'\left(\frac{\ell_0(\theta) - r}{\varepsilon}\right), \\[2mm] B(r, \theta) &= \partial_\theta \Theta(r, \theta) &= 1 - (\alpha - \alpha_0)\frac{1}{\varepsilon}\chi'\left(\frac{\alpha_0 - \theta}{\varepsilon}\right), \\[2mm] C(r, \theta) &= \partial_\theta R(r, \theta) &= (\ell'_\beta(\theta) - \ell'_0(\theta))\chi\left(\frac{\ell_0(\theta) - r}{\varepsilon}\right) + (\ell_\beta(\theta) - \ell_0(\theta))\frac{\ell'_0(\theta)}{\varepsilon}\chi'\left(\frac{\ell_0(\theta) - r}{\varepsilon}\right). \end{cases}$$

Observe that Θ does not depend on r, so that $\partial_r\Theta(r,\theta) = 0$. From these expressions, we derive the following lemma.

Lemma 4. *For any $\varepsilon > 0$ there exists ρ_ε such that Φ is a smooth diffeomorphism from T_0 onto T as soon as $d(T, T_0) < \rho_\varepsilon$.*

Proof. We first choose α close enough to α_0 so that B is uniformly bounded below by some positive (small) constant. It follows that Θ is a smooth diffeomorphism from $[0, \alpha_0]$ onto $[0, \alpha]$ and that $|\Theta(\theta) - \theta| = O(|\alpha - \alpha_0|)$ uniformly. By definition $(\beta, \theta) \mapsto L_\beta(\theta)$ is smooth so that, if ρ is small enough, then A is also bounded below by some positive constant. It follows that Φ is a smooth bijective mapping from T_0 onto T. Restricting ρ again if needed, we can ensure the Jacobian matrix to be always invertible and this proves the claim. $\square$

Using Φ, T is then parametrized by T_0. The pulled-back metric $\Phi^*(dr^2 + \sin^2 rd\theta^2)$ is now represented by the matrix $\mathbb{G}$ defined by

$$\mathbb{G}(r,\theta) = {}^t\mathrm{Jac}\,\Phi_{|(r,\theta)} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 R(r,\theta) \end{pmatrix} \mathrm{Jac}\,\Phi_{|(r,\theta)}.$$

It is convenient to set $D(r,\theta) = \sin R(r,\theta)$ and to define the (Euclidean) gradient

$$\nabla u = \begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix}.$$

With these notations, the Dirichlet quadratic form (10) now reads

(13)
$$\begin{aligned} q(u\,;\,\alpha,\beta) &= \int_{T_0} {}^t\nabla u(r,\theta)\mathbb{G}^{-1}(r,\theta)\nabla u(r,\theta)\,ABD(r,\theta)drd\theta \\ &= \int_{T_0} \left[(C^2 + D^2B^2)(\partial_r u)^2 - 2AC\partial_r u\partial_\theta u + A^2(\partial_\theta u)^2\right] \frac{drd\theta}{ABD} \end{aligned}$$

and the L^2 scalar product (11) reads

(14)
$$n(u\,;\,\alpha,\beta) = \int_{T_0} u^2 ABD\,drd\theta.$$

Using the definitions, we first observe that the quadratic forms $q(\cdot\,;\,\alpha,\beta)$ are uniformly equivalent for (α,β) in a small neighbourhood of (α_0,β_0), and similarly for $n(\cdot\,;\,\alpha,\beta)$. The completion procedure that is used to define the Friedrichs extension thus yields a domain that does not depend on (α,β) and thus coincides with $H_0^1(T_0)$.

Moreover, for any fixed $u \in H_0^1(T_0)$, the functions $(\alpha,\beta) \mapsto q(u\,;\,\alpha,\beta)$ and $(\alpha,\beta) \mapsto n(u\,;\,\alpha,\beta)$ are analytic for (α,β) close to (α_0,β_0). It follows that analytic perturbation theory applies and yields the following properties:

- If λ_0 is a simple eigenvalue of T_0, then there exists $\delta > 0$ and a neighbourhood of T_0, such that, in this neighbourhood, there is a unique eigenvalue of T in $(\lambda_0 - \delta, \lambda_0 + \delta)$ and this eigenvalue depends analytically on (α,β).
- For any (real-)analytic curve $t \mapsto (\alpha(t), \beta(t))$ on some interval I, there exists a collection $\left(t \mapsto E_i(t)\right)_{i \geqslant 1}$ of real-analytic functions that exhaust the spectrum of $T_t = T(\alpha(t), \beta(t))$. Such a function is called an analytic eigenvalue branch and there also exist corresponding analytic eigenfunction branches $t \mapsto u_i(t)$.

- The derivatives of the eigenbranches are given by the Feynman-Hellmann formula (see [17] or [16] prop. 4.6 for a proof in a similar setting): for an analytic eigenbranch $t \mapsto (E(t), u(t))$, we have

$$(15) \qquad \forall t \in I, \quad \dot{E}(t)\|u(t)\|^2 \,=\, [\dot{q}_t - E(t)\dot{n}_t]\,(u(t)),$$

in which the dot denotes the derivative with respect to t. This formula is obtained by differentiating (12); specifically, we first differentiate $q_t(v)$ and $n_t(v)$ with a fixed v and then evaluate $v = u$.

If λ_0 is an eigenvalue of T_0 of multiplicity m, it follows by standard min-max arguments that, for δ small enough, there exist exactly m eigenvalues of T in $(\lambda_0 - \delta, \lambda_0 + \delta)$ in a small neighbourhood of T_0. In a nutshell, analytic perturbation theory says that, along any curve that is real-analytic, it is possible to label these m eigenvalues so as to have analytic functions. There are, however, two problems remaining. First, the labeling does not preserve the order of eigenvalues: analytic eigenbranches will typically cross at T_0. Then, it is usually not possible to define eigenbranches that would be analytic for (α, β) in a neighbourhood: the labeling depends on the analytic curve that is chosen and cannot be done consistently in all directions. Of course both problems only arise for multiple eigenvalues.

We have expressed the derivatives of the eigenvalue branch using the corresponding eigenfunction branch. For the reasons given in the preceding paragraph, it is convenient to give a way to recover the derivatives without knowing a priori the eigenfunction branch. This is obtained by the following procedure.

Let λ_0 be an eigenvalue of T_0 and $\mathcal{E}_0$ the corresponding eigenspace. The derivatives of all the eigenbranches that coincide with λ_0 at $t = 0$ are exactly the eigenvalues of the quadratic form $\dot{q} - \lambda_0 \dot{n}$, restricted to $\mathcal{E}_0$ and relative to the scalar product n. Observe that using (9) we can write

$$(16) \qquad \dot{q} - \lambda_0 \dot{n} \,=\, \dot{\alpha}\big(\partial_\alpha q - \lambda_0 \partial_\alpha n\big) \,+\, \dot{\beta}\big(\partial_\beta q - \lambda_0 \partial_\beta n\big),$$

so that, although we may not have differentiability of the eigenvalues, still, it is enough to know the partial derivatives $\partial_\alpha q - \lambda_0 \partial_\alpha n$ and $\partial_\beta q - \lambda_0 \partial_\beta n$ to compute the derivatives of the eigenbranches in any direction.

2.3. **A Hadamard variational formula.** The formulas in the preceding section express the derivative of the eigenbranches using integrals over the whole domain T_0, of some quadratic expressions in u, $\partial_r u$, $\partial_\theta u$, see (15) and (16). Hadamard variational formulas use integrals only on the boundary of the domain, see Theorems 3 and 4 below. Since the latter are of independent interest and give slightly simpler computations in the end, we explain here how to derive them. This derivation is made possible by computing $\dot{q} - \lambda_0 \dot{n}$ for fixed ε and then letting our parameter ε go to 0. More precisely, for any ε, we define the two quadratic forms (see (16))

$$D_\alpha^\varepsilon = \partial_\alpha q - \lambda_0 \partial_\alpha n \text{ and } D_\beta^\varepsilon = \partial_\beta q - \lambda_0 \partial_\beta n,$$

that are obtained from (13) and (14), where recall that the dependence on ε comes from the diffeomorphism $\Phi_{\alpha,\beta}^{(\varepsilon)}$.

Proposition 5. *Let λ_0 be an eigenvalue of T_0 and $\mathcal{E}_0$ the corresponding eigenspace. For any $u \in \mathcal{E}_0$,*

$$\lim_{\varepsilon \to 0} D_\alpha^\varepsilon(u) = -\int_0^{\ell(\alpha)} \frac{|\partial_\theta u(r, \alpha)|^2}{\sin r}\, dr,$$

$$\lim_{\varepsilon \to 0} D_\beta^\varepsilon(u) = -\int_0^{\alpha_0} \left[|\partial_r u(L_{\beta_0}(\theta), \theta)|^2 + \frac{|\partial_\theta u(L_{\beta_0}(\theta), \theta)|^2}{\sin^2 L_{\beta_0}(\theta)} \right] (\partial_\beta L)_{\beta_0}(\theta) \sin L_{\beta_0}(\theta)\, d\theta.$$

Proof. Since $u \in H_0^1$, the expressions $D_\alpha^\varepsilon(u)$ and $D_\beta^\varepsilon(u)$ can be obtained by differentiating under the integral sign the expressions given in (13) and (14).

Thus, for D_α^ε, we need to compute $F(r, \theta\,;\,\alpha_0, \beta_0)$, $\partial_\alpha F(r, \theta\,;\,\alpha_0, \beta_0)$ for $F = A, B, C, D$. Of these four quantities, only $\partial_\alpha B$ does not vanish identically and after a somewhat lengthy but straightforward computation, we obtain

$$D_\alpha^\varepsilon(u) = \int_{T_0} \frac{1}{\varepsilon} \chi'\Big(\frac{\alpha_0 - \theta}{\varepsilon}\Big) \left[\frac{|\partial_\theta u|^2}{\sin^2 r} + \lambda_0 |u|^2 \right] \sin r\, dr d\theta.$$

We now let ε go to 0. When tested against sufficiently well-behaved functions, $\frac{1}{\varepsilon} \chi'(\alpha_0 - \theta)$ converges to the integration on the side

$$\big\{ (r, \alpha_0) : \quad r \in [0, L_{\beta_0}(\alpha_0)] \big\}.$$

We will provide, in Appendix B, all the necessary estimates showing that this limit is justified when u is an eigenfunction. We then obtain

$$\lim_{\varepsilon \to 0} D_\alpha^\varepsilon(u) = -\int_0^{\ell(\alpha)} \frac{|\partial_\theta u(r, \alpha)|^2}{\sin r}\, dr.$$

The second term vanishes since u satisfies the Dirichlet boundary condition.

For D_β^ε we follow the same strategy, computing now the derivatives with respect to β, still evaluated at (α_0, β_0). We find:

$$D_\beta^\varepsilon(u) = \int_{T_0} \partial_\beta L_{\beta_0} |\partial_r u|^2 \left[(\cos r)\chi\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) + \frac{\sin r}{\varepsilon}\chi'\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) \right] dr d\theta$$

$$- 2\int_{T_0} \frac{\partial_r u \partial_\theta u}{\sin r} \left[(\partial_\beta L'_{\beta_0})\chi\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) + \partial_\beta L_{\beta_0} \frac{L'_{\beta_0}}{\varepsilon}\chi'\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) \right] dr d\theta$$

$$- \int_{\beta_0} \frac{|\partial_\theta u|^2}{\sin^2 r} \left[(\cos r)\chi\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) + \frac{\sin r}{\varepsilon}\chi'\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) \right] dr d\theta$$

$$- \lambda_0 \int_{T_0} \partial_\beta L_{\beta_0} |u|^2 \left[(\cos r)\chi\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) - \frac{\sin r}{\varepsilon}\chi'\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) \right] dr d\theta.$$

As above, we will give in Appendix B the needed estimates to prove that the terms with $\chi\big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\big)$ converge to 0 and the terms with $\frac{1}{\varepsilon}\chi'\big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\big)$ converge to a boundary integral over the side $\big\{ (L_{\beta_0}(\theta), \theta) : \theta \in [0, \alpha_0] \big\}$. Since u satisfies the Dirichlet boundary condition, u vanishes on the latter side. Denoting by γ the parametrization $\theta \mapsto (L_{\beta_0}(\theta), \theta)$, we obtain

$$\lim_{\varepsilon \to 0} D_\beta^\varepsilon(u) =$$
$$-\int_0^{\alpha_0} \left[|\partial_r u \circ \gamma|^2 - 2L'_{\beta_0}(\theta)\frac{(\partial_r u \partial_\theta u) \circ \gamma}{\sin^2 L_{\beta_0}(\theta)} - \frac{|\partial_\beta u \circ \gamma|^2}{\sin^2 L_{\beta_0}(\theta)} \right] (\partial_\beta L)_{\beta_0}(\theta) \sin(L_{\beta_0}(\theta))\, d\theta.$$

This expression can be simplified further by observing that Dirichlet boundary condition implies that $\left(L'_{\beta_0} \partial_r u + \partial_\theta u \right) \circ \gamma = 0$. Finally, we obtain

$$\lim_{\varepsilon \to 0} D_\beta^\varepsilon(u) = -\int_0^{\alpha_0} \left[|\partial_r u \circ \gamma|^2 + \frac{\partial_\theta u \circ \gamma|^2}{\sin^2 L_{\beta_0}(\theta)} \right] \partial_\beta L(\theta\,; \alpha_0, \beta_0) \sin L_{\beta_0}(\theta)\, d\theta. \qquad \square$$

Remark 3. *The estimates of the appendix are needed to properly prove the convergence when ε goes to zero for any eigenvalue branch at any triangle T_0. The proof of Theorem 2 needs this computation only for the first three eigenvalues and at the equirectangle triangle. In the latter case, we have an explicit expression for the eigenfunction so that the convergence can be proved directly without referring to the general Sobolev theory on singular domains.*

Combining the latter proposition and the results of analytic perturbation theory that we have recalled in the previous section, we obtain the following two theorems.

Theorem 3. *Let $T_0 \in \mathcal{M}$ and λ_0 be a simple eigenvalue of the Dirichlet spherical Laplace operator of T_0. There exist $\delta > 0$ and a neighbourhood $U \subset \mathcal{M}$ of T_0 such that:*

- *Any triangle T in U has a unique eigenvalue λ in $(\lambda_0 - \delta, \lambda_0 + \delta)$.*
- *The mapping $T \mapsto \lambda(T)$ is real-analytic on U.*
- *For any $T = T(\alpha, \beta) \in U$, we have*

$$\partial_\alpha \lambda(T) = -\int_0^{L_\beta(\alpha)} \frac{|\partial_\theta u(r, \alpha)|^2}{\sin r}\, dr,$$

$$\partial_\beta \lambda(T) = -\int_0^\alpha \left[|\partial_r u(L_\beta(\theta), \theta)|^2 + \frac{|\partial_\theta u(L_\beta(\theta), \theta)|^2}{\sin^2 L_\beta(\theta)} \right] (\partial_\beta L)_\beta(\theta) \sin L_\beta(\theta)\, d\theta,$$

where u is a $L^2(T)$ normalized eigenfunction.

Theorem 4. *For t in an interval I, let $t \mapsto T(\alpha(t), \beta(t)) = T_t$ be an analytic family of spherical triangles and λ_0 be an eigenvalue of multiplicity m of T_0. Then there exist m analytic functions $(E_k)_{1 \leqslant k \leqslant m}$ defined on I such that:*

(i) *There exist $\delta_0, t_0 > 0$ such that, for any $t \in (-t_0, t_0)$ and any eigenvalue λ in $\mathrm{Spec}\,(T_t) \cap (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$, the multiplicity of λ is the number of k such that $E_k(t) = \lambda$.*

(ii) *The derivatives $\dot{E}_k(0)$ are the eigenvalues of the quadratic form*

$$(17) \qquad u \mapsto -\dot{\alpha}(0) \int_0^{L_\beta(\alpha)} \frac{|\partial_\theta u(r, \alpha)|^2}{\sin r}\, dr$$

$$-\dot{\beta}(0) \int_0^\alpha \left[|\partial_r u(L_\beta(\theta), \theta)|^2 + \frac{\partial_\theta u(L_\beta(\theta), \theta)|^2}{\sin^2 L_\beta(\theta)} \right] (\partial_\beta L)_\beta(\theta) \sin L_\beta(\theta)\, d\theta,$$

restricted to the eigenspaces of λ_0 and relatively to the L^2 norm on T.

Proof. The proof of the two theorems follows the same line. First we fix some $\varepsilon > 0$. The first statements, in particular the existence of U (or t_0), follow from the previous section using the family of diffeomorphisms Φ^ε. It remains to compute the derivatives. For this, we pick a triangle T in the neighbourhood U, and we write, for each derivative and each $\varepsilon > 0$ the formula that is obtained using Φ^ε (with now T as the starting point). Using

analyticity, the eigenvalue branches that we obtain do not depend on ε. So for each ε, the formula for the derivative of simple eigenbranches gives the same value. For a multiple eigenvalue, the quadratic form that gives the derivatives has the same eigenvalues. We can thus let ε go to zero and and Proposition 5 then yields the result. $\qquad\square$

2.4. **The first eigenvalue on $\mathcal{M}$.** In this section, we describe the first eigenvalue λ_1 as a function on the set of spherical triangles $\mathcal{M}$.

It is well known that the first eigenvalue of a domain in a Riemannian manifold is always simple. It then follows that the eigenvalue λ_1 depends analytically on (α, β) in $(0, \pi) \times (0, \pi)$. We now make a list of several facts that help us understand the level sets of λ_1.

(i) *Symmetry:* the symmetry with respect to the median hyperplane of $[A_*, B_*]$ in the sphere exchanges $T(\alpha, \beta)$ and $T(\beta, \alpha)$. The function λ_1 is thus symmetric with respect to $(\alpha, \beta) \leftrightarrow (\beta, \alpha)$.

(ii) *Monotonicity:* if $\alpha' \geqslant \alpha$ and $\beta' \geqslant \beta$, then the triangle $T(\alpha, \beta)$ is a subset of $T(\alpha', \beta')$. Using the min-max principle, Dirichlet eigenvalues are shown to be decreasing relative to the inclusion of domains. We thus infer:

$$\alpha' \geqslant \alpha \ \text{ and } \ \beta' \geqslant \beta \ \implies \ \lambda_1(\alpha', \beta') \leqslant \lambda_1(\alpha, \beta).$$

Since the first Dirichlet eigenvalue of the hemisphere is 2, we also get that

$$\forall T \in \mathcal{M}, \ \lambda_1(T) \geqslant 2.$$

(iii) *Regularity:*

$$\forall (\alpha, \beta) \in (0, \pi) \times (0, \pi), \ \ \partial_\alpha \lambda_1(\alpha, \beta) < 0, \ \text{ and } \ \partial_\beta \lambda_1(\alpha, \beta) < 0.$$

Let us prove that $\partial_\alpha \lambda_1 \neq 0$ by contradiction. If this derivative vanishes then the integral formula of Theorem 3 implies that $\partial_\theta u$ vanishes on one side of the triangle. If we reflect the triangle across this side, we obtain a rhombus to which we extend u by 0. We denote by $\widetilde{u}$ this extension and we test $(\Delta - \lambda)\widetilde{u}$ against a smooth function with compact support in the rhombus. Using integration by parts (Green's formula) inside and outside the original triangle, we obtain an integral over the side across which we have reflected. This integral vanishes because u and $\partial_\theta u$ vanish on that side. This proves that $\widetilde{u}$ is an eigenfunction (with the same eigenvalue) of the Dirichlet Laplace operator in the rhombus. This violates the principle of analytic continuation for eigenfunctions. By symmetry, the derivative with respect to β cannot vanish. The monotonicity gives the sign.

(iv) *Behaviour near the boundary:* If α or β goes to 0, then λ_1 goes to infinity. This is a general fact about shrinking domains with Dirichlet boundary condition. For instance, here, we could use the minmax principle to compare with a spherical angular sector whose angle goes to 0.

If α goes to π and β goes to $\beta_0 \in (0, \pi]$, the first eigenvalue $\lambda_1(\alpha, \beta)$ converges to the first eigenvalue of the $\mathcal{D}_{\beta_0}$, the digon of opening angle β_0 (see Figure 4, left). Indeed, the family of spectral problems is continuous up to $(0, \pi] \times (0, \pi]$.

(v) *Behaviour on the boundary:* All the computations we made are still valid for $\alpha = \pi$ and varying β. It follows that the mapping μ which, to an angle $\beta \in (0, \pi]$,

FIGURE 5. All these triangles have first Dirichlet eigenvalue $\lambda_1 = 12$. The first domain is the equirectangle triangle T_* and the last domain is the digon of opening $\frac{\pi}{3}$. The second eigenvalue is strictly decreasing along this path of triangles, going from 30 to 20.

assigns the first Dirichlet eigenvalue of the digon of opening angle β, is an analytic, decreasing diffeomorphism from $(0, \pi)$ onto $(2, \infty)$ that extends continuously at π.

Remark 4. *Observe that in* [23] *(see also* [25, 26]*), it is proved that the spectrum of the digon of angle β can be explicitly computed. This computation yields $\mu(\beta) = \frac{\pi}{\beta}(\frac{\pi}{\beta} + 1)$ which obviously satisfies all claimed features.*

These properties allow us to prove the following proposition, which gives a reasonably complete understanding of how λ_1 behaves as a function on $\mathcal{M}$. See Figure 5 for an illustration.

Proposition 6. *For any $c \in (2, \infty)$, there exists $\alpha_c \in (0, \pi)$ such that the first eigenvalue of $\mathcal{D}_{\alpha_c}$ is c. The level set $\lambda_1^{-1}\{c\} \subset \mathcal{M}$ is an analytic curve that can be globally parametrized by $\alpha \in (\alpha_c, \pi)$. More precisely, there exists an analytic function $\mathcal{B}_c$ from (α_c, π) such that*

$$\lambda_1(T(\alpha, \beta)) = c \iff \beta = \mathcal{B}_c(\alpha).$$

The function $\mathcal{B}_c$ is decreasing, extends continuously to $[\alpha_c, \pi]$ by $\mathcal{B}_c(\alpha_c) = \pi$ and $\mathcal{B}_c(\pi) = \alpha_c$ and $\mathcal{B}_c'(\frac{\pi}{2}) = -1$.

Proof. The first statement follows from the known behaviour of the first eigenvalue of digons, see Remark 4. Moreover, using monotonicity in β, for each $\alpha > \alpha_c$ the mapping $\beta \mapsto \lambda_1(\alpha, \beta)$ is a decreasing diffeomorphism onto $(\mu(\alpha), \infty)$. Since this interval contains c, there is a unique β that we denote by $\mathcal{B}_c(\alpha)$ such that $\lambda_1(\alpha, \beta) = c$. The fact that $\mathcal{B}_c(\alpha)$ is analytic then follows from the implicit function theorem. The remaining statements follow from the behaviour of λ_1 near and on the boundary and the symmetry. $\square$

3. PROOF OF THEOREM 2

Proof. Let T_* be the equirectangle triangle with vertices A_*, B_*, C_*, see Figure 3. It corresponds to $\alpha = \beta = \frac{\pi}{2}$. We have $\lambda_1(T_0) = 12$. In the sequel, we set $\mathcal{B} = \mathcal{B}_c$ for $c = 12$. Using the formula in [23, Proposition 1.1] for the eigenvalues of the digons, we obtain that the function $\mathcal{B}$ is defined on $[\frac{\pi}{3}, \pi]$, analytic in $(\frac{\pi}{3}, \pi)$ and

$$\forall \alpha \in \left[\frac{\pi}{3}, \pi\right], \quad \lambda_1(\alpha, \mathcal{B}(\alpha)) = 12.$$

Let I be a small interval around 0, for $t \in I$, we define

$$\alpha(t) = \frac{\pi}{2} + t, \quad \beta(t) = \mathcal{B} \circ \alpha(t)$$

and set $T_t = T(\alpha_t, \beta_t)$. The curve $t \mapsto T_t$ is real analytic and, using the properties of $\mathcal{B}$, we know that the spectrum of T_t is organized into analytic branches. Since the second eigenvalue of T_0 is of multiplicity 2, we can restrict I so that there exists three analytic functions E_1, E_2, E_3 such that

$$\forall t \in I, \ \lambda_1(t) = E_1(t) = 12, \quad \lambda_2(t) = \min\{E_2(t), E_3(t)\}, \quad \lambda_3(t) = \max\{E_2(t), E_3(t)\}.$$

Let $\mathcal{E}$ be the (two-dimensional) eigenspace corresponding to $\lambda_2(0)$. We make the formulas (17) explicit, using in particular that $(\partial_\beta L)_{\frac{\pi}{2}}(\theta) = \sin\theta$ and that $\dot\alpha(0) = 1$ and $\dot\beta(0) = -1$. Thus, the derivatives of E_2 and E_3 are given by the eigenvalues of the quadratic form

$$d(u) = -\int_0^{\frac{\pi}{2}} \frac{|\partial_\theta u(r, \frac{\pi}{2})|^2}{\sin r} \, dr + \int_0^{\frac{\pi}{2}} |\partial_r u(\frac{\pi}{2}, \theta)|^2 \sin\theta \, d\theta$$

with respect to the L^2 norm. We define on T_0 the functions u_1 and u_2 by

$$\begin{cases} u_1(r, \theta) &= \sqrt{\frac{1155}{8\pi}} \left(3\cos^5 r - 4\cos^3 r + \cos r\right) \sin 2\theta, \\ u_2(r, \theta) &= \sqrt{\frac{3465}{32\pi}} \cos r \sin^4 r \sin 4\theta, \end{cases}$$

that form a L^2-orthonormal basis of $\mathcal{E}$ (see in [23, Cor. 2.1] for instance).

We compute

$$\int_0^{\frac{\pi}{2}} \frac{|\partial_\theta u_1(r, \frac{\pi}{2})|^2}{\sin r} \, dr = \frac{1155}{8\pi} \int_0^{\frac{\pi}{2}} \frac{4\left(3\cos^5 r - 4\cos^3 r + \cos r\right)^2}{\sin r} \, dr = \frac{44}{\pi},$$

$$\int_0^{\frac{\pi}{2}} \frac{|\partial_\theta u_2(r, \frac{\pi}{2})|^2}{\sin r} \, dr = \frac{3465}{32\pi} \int_0^{\frac{\pi}{2}} \frac{16\cos^2 r \sin^8 r}{\sin r} \, dr = \frac{44}{\pi},$$

$$\int_0^{\frac{\pi}{2}} \frac{\partial_\theta u_1(r, \frac{\pi}{2}) \partial_\theta u_2(r, \frac{\pi}{2})}{\sin r} \, dr = C \int_0^{\frac{\pi}{2}} \frac{\left(3\cos^5 r - 4\cos^3 r + \cos r\right) \cos r \sin^4 r}{\sin r} \, dr = 0,$$

where C is some numerical constant that we do not need to write down explicitly, and

$$\int_0^{\frac{\pi}{2}} |\partial_r u_1(\frac{\pi}{2}, \theta)|^2 \sin\theta \, d\theta = \frac{1155}{8\pi} \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \sin\theta d\theta = \frac{77}{\pi},$$

$$\int_0^{\frac{\pi}{2}} |\partial_r u_2(\frac{\pi}{2}, \theta)|^2 \sin\theta \, d\theta = \frac{3465}{32\pi} \int_0^{\frac{\pi}{2}} \sin^2(4\theta) \sin\theta \, d\theta = \frac{55}{\pi},$$

$$\int_0^{\frac{\pi}{2}} \partial_r u_1(\frac{\pi}{2}, \theta) \partial_r u_2(\frac{\pi}{2}, \theta) \sin\theta \, d\theta = \frac{1155\sqrt{3}}{16\pi} \int_0^{\frac{\pi}{2}} \sin(2\theta) \sin(4\theta) \sin\theta \, d\theta = -11\sqrt{3}.$$

We obtain that the matrix that represents d in the basis (u_1, u_2) of $\mathcal{E}$ is

$$D = -\frac{11}{\pi} \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix}.$$

Since (u_1, u_2) is a L^2 orthonormal basis of $\mathcal{E}$, the derivatives $\dot{E}_2$ and $\dot{E}_3$ are the eigenvalues of D that we compute to be $\pm\frac{22\sqrt{3}}{\pi}$. Finally, we obtain, for t close to 0,

$$
\begin{cases}
E_2(t) &= 12 + t \cdot \frac{22\sqrt{3}}{\pi} + o(t), \\
E_3(t) &= 12 - t \cdot \frac{22\sqrt{3}}{\pi} + o(t), \\
\lambda_2(t) &= 30 - |t| \cdot \frac{22\sqrt{3}}{\pi} + o(t), \\
\lambda_3(t) &= 30 + |t| \cdot \frac{22\sqrt{3}}{\pi} + o(t),
\end{cases}
$$

and the proof is complete. $\hfill\square$

Remark 5. *Using the computations above, the quadratic form that is given by equation* (17) *at the equirectangle triangle when expressed in the basis* (u_1, u_2) *yields the same expression as (3.11) in* [23]. *We could have use analyticity to claim that the formula loc. cit., which is proved for variations of triangles of diameter* $\frac{\pi}{2}$ *(using different variational —Feynman-Hellmann type—formulas), still holds for any variation. We have found it interesting to write down the Hadamard variational formula so as to have a (slightly) different proof.*

Remark 6. *This perturbation approach can be implemented starting from any initial triangle* T_*, *and, basically, we only need to show that the ratio* $\frac{\lambda_2}{\lambda_1}$ *cannot be always rational. Although this seems a rather weak statement, our proof requires some rather precise knowledge of the eigenfunctions. This explains our choice of the equirectangle triangle.*

Appendix A. Regularity of eigenfunctions of a triangle

The aim of this appendix is to provide the necessary estimates that allow to pass to the limit $\varepsilon \to 0$ in order to obtain the Hadamard variational formulas. All the results can be extracted from the literature on elliptic problems in domains with corners (see [15, 19] for instance). We have chosen to give some ideas of the proof so as to have a self-contained presentation.

We fix a spherical triangle $T(\alpha, \beta)$ for some $(\alpha, \beta) \in (0, \pi)^2$. Explicitly, it is the domain as in (9) equipped with the metric $g = dr^2 + \sin^2 r d\theta^2$. We may see T as a subset of the plane $\mathbb{R}^2$ equipped with the coordinates $x = r\cos\theta$ and $y = r\sin\theta$. The metric g is then uniformly equivalent to the Euclidean metric $dx^2 + dy^2$.

We recall the definition of the usual Sobolev spaces, for $k \in \mathbb{N}$ and using the common multiindex notation:

$$H^k(T) = \{u \in L^2(dxdy): \quad \forall\alpha, \ |\alpha| \leqslant k, \ \partial^\alpha u \in L^2(dxdy)\},$$

$$\forall u \in H^k(T), \ \|u\|_{H^k}^2 = \sum_{|\alpha|\leqslant k} \|\partial^\alpha u\|_{L^2(dxdy)}.$$

We also recall that $H_0^1(T)$ is the completion of $C_0^\infty(T)$ with respect to the H^1 norm.

Writing the partial derivatives ∂_x and ∂_y in polar coordinates, we see that the Sobolev space $H^1(T)$ can alternatively be defined as follows:

$$H^1(T) = \{u \in L^2(T, \sin r drd\theta): \quad \partial_r u \text{ and } \frac{1}{\sin r}\partial_\theta u \in L^2(T, \sin r drd\theta)\}.$$

We observe that the set $H_0^1(T)$ is defined in Section 2.1 as the completion of $C_0^\infty(T)$ with respect to the Dirichlet energy quadratic form q. We see here that it coincides with the usual one.

The norm that is associated with the spherical Laplace operator is

$$\forall u \in C_0^\infty(T), \quad \|u\|_\Delta^2 = \|u\|_{L^2(\sin r dr d\theta)}^2 + \|\Delta u\|_{L^2(\sin r dr d\theta)}^2.$$

By standard ellipticity estimates, this norm is equivalent to the H^2 norm.

Lemma 7. *The space $C_0^\infty(T)$ is dense in $\mathrm{dom}\,\Delta$ with respect to the $\|\cdot\|_\Delta$ norm.*

Proof. Let $u \in \mathrm{dom}\,\Delta$ that is orthogonal to $C_0^\infty(T)$ with respect to the Hilbertian norm $\|\cdot\|_\Delta$. Double T to form a surface $\widetilde{T}$ that is a sphere equipped with a singular spherical metric that has three cone points A, B, C at which the total angle is (strictly) less than 2π. Extend u a function $\widetilde{u}$ on $\widetilde{T}$ that is odd with respect to the natural involution of $\widetilde{T}$. Using Green's formula, by construction, we have

$$\forall \phi \in C_0^\infty(\widetilde{T} \setminus \{A, B, C\}), \quad \langle \widetilde{u}, (\Delta + 1)\phi \rangle = 0.$$

It follows that, in the distributional sense, $(\Delta + 1)\widetilde{u}$ is supported at A, B, C. The latter can be explicitly described using polar coordinates and it turns out that, since the total angle is less than 2π, $\widetilde{u}$ must also be even. Thus, the function $\widetilde{u}$ vanishes and thus also u. $\qquad\square$

The latter lemma gives $\mathrm{dom}\,\Delta = \mathring{W}_2^2(T)$ in the notations of [15], and Theorem 1.4.4.4 *loc. cit.* applies to give the following corollary.

Corollary 8. *For any $u \in \mathrm{dom}\,\Delta$, $(\sin r)^{-2}u$, $(\sin r)^{-1}\partial_r u$ and $(\sin r)^{-2}\partial_\theta u$ belong to $L^2(\sin r dr d\theta)$.*

Proof. Theorem 1.4.4.4 of [15] gives $r^{-2}u$, $r^{-1}\partial_x u$ and $r^{-1}\partial_y u$ in $L^2(T, dxdy)$. The claim follows using

$$(18) \qquad\qquad r\partial_r u = x\partial_x u + y\partial_y u \quad \text{and} \quad \partial_\theta u = x\partial_y u - y\partial_x u. \qquad\square$$

Finally, we obtain:

Proposition 9. *Let u be an element of the domain of the Dirichlet Laplace operator. Then u, $\partial_r u$ and $\frac{1}{\sin r}\partial_\theta u$ belong to $H^1(T)$.*

Proof. Since $u \in \mathrm{dom}\,\Delta$, then all derivatives with respect to (x, y) of order at most 2 are in L^2. We compute $\partial_r^2 u$ and $\frac{1}{\sin r}\partial_\theta\partial_r u$ in Cartesian coordinates using the relations (18), and we check by inspection that these are L^2. We do the same for $\partial_r \frac{1}{\sin r}\partial_\theta u$ and $\frac{1}{\sin^2 r}\partial_\theta^2 u$. $\qquad\square$

APPENDIX B. APPLICATION TO THE HADAMARD FORMULA

In this section, we use the regularity of eigenfunctions to prove that the limit $\varepsilon \to 0$ that is used in the proof of Hadamard variational formula is justified (see the proof of Proposition 5). As in Remark 3, we emphasize here that it is actually not needed to prove Theorem 2 since we would only need to consider the first three eigenfunctions of the equirectangle triangle for which explicit bounds can be given.

We begin by addressing the limit of $D_\beta^\varepsilon(u)$ since it gives simpler computations; the global strategy being the same. We first observe that in all terms, the χ or χ' factor cuts-off near $r = L_{\beta_0}(r)$. Using that $u \in H_0^1(T)$, all terms with the factor χ converge to 0 by integration of a L^1 function on a shrinking domain. It remains to address the terms with a χ'. All these terms can be written under the expression

$$\mathbb{B}_\varepsilon(h) = \int_0^{\alpha_0} \int_0^{\ell(\theta)} \frac{1}{\varepsilon} \chi'\Big(\frac{L_{\beta_0}(\theta) - r}{\varepsilon}\Big) h(r, \theta) \, dr d\theta,$$

in which h is obtained as a product of a smooth function (away of $r = 0$) times two first order derivatives of u. Since $u \in \mathrm{dom}\, T$, the partial derivatives $\partial_r u$ and $\partial_\theta u$ are in $H^1(T \cap \{r \geqslant r_0 > 0\})$. It follows that h belongs to $W_1^1(T)$: the set of L^1 functions whose gradient is also L^1.

Changing variables, we have

$$\mathbb{B}_\varepsilon(h) = \int_0^1 \int_0^{\alpha_0} \chi'(\rho) h(L_{\beta_0}(\theta) - \varepsilon\rho, \theta) d\rho d\theta.$$

Since $\int \chi'(\rho) d\rho = -1$, we see that

$$\mathbb{B}_\varepsilon(h) + \int_0^{\alpha_0} h(L_\beta(\theta), \theta) \, d\theta = \int_0^1 \int_0^{\alpha_0} \chi'(\rho) \left[h(L_{\beta_0}(\theta) - \varepsilon\rho, \theta) - h(L_{\beta_0}(\theta), \theta) \right] d\rho d\theta.$$

We define the mapping Φ on $[0,1] \times [0,1] \times [0, \alpha_0]$ by $(\rho, s, \theta) \mapsto (\rho, L_{\beta_0}(\theta) - s\varepsilon\rho, \theta)$. For any fixed θ and ρ, $s \mapsto \Phi(\rho, s, \theta)$ parametrizes a curve in T and since $u \in W_1^1$, we can write

$$h(L_{\beta_0}(\theta) - \varepsilon\rho, \theta) - h(L_{\beta_0}(\theta), \theta) = -\int_0^1 \varepsilon\rho \partial_r h \circ \Phi(\rho, s, \theta) \, ds.$$

This leads us to consider the integral

$$R_\varepsilon = \int_0^1 \int_0^1 \int_0^{\alpha_0} \left| \chi'(\rho) \varepsilon\rho \partial_r h \circ \Phi(\rho, s, \theta) \right| ds d\rho d\theta.$$

The mapping Φ is obviously smooth and injective, and since its Jacobian determinant is $-\varepsilon\rho$, the change of variables is legitimate. Using that ρ' has support in $[\frac{1}{3}, \frac{2}{3}]$, we obtain

$$R_\varepsilon = \int_{U_\varepsilon} \left| \chi'(\rho) \partial_r h(r, \theta) \right| d\rho dr d\theta,$$

where U_ε is the image of $[\frac{1}{3}, \frac{2}{3}] \times [0,1] \times [0, \alpha]$ under Φ. We define the set $V_\varepsilon \subset T$ such that

$$\{1\} \times V_\varepsilon = \Phi\{1\} \times [0,1] \times [0, \alpha_0],$$

it is straightforward that $U_\varepsilon \subset [0,1] \times V_\varepsilon$ and that U_ε is a shrinking neighbourhood in T of the side $\{(L(\theta), \theta) : \theta \in [0, \alpha_0]\}$. Since $\partial_r h \in L^1(T)$, we obtain that R_ε goes to 0 with ε and, hence,

$$\lim_{\varepsilon \to 0} \mathbb{B}_\varepsilon(h) = -\int_0^{\alpha_0} h(L_\beta(\theta), \theta) \, d\theta.$$

Applying this result to the different terms in $D_\beta^\varepsilon(u)$ yields the Hadamard formula for $\partial_\beta E$.

We follow the same strategy to study the limit of $D_\alpha^\varepsilon(u)$. We have two terms that can be brought to the expression

$$\mathbb{A}_\varepsilon(h) = \int_0^{\alpha_0} \int_0^{L_{\beta_0}(\theta)} \frac{1}{\varepsilon} \chi'\left(\frac{\alpha_0 - \theta}{\varepsilon}\right) h(r,\theta)\, r dr d\theta,$$

with $h \in W_1^1(T)$. We first perform a change of variables that fixes the domain of integration. For (ρ, t) in $[0, L_{\beta_0}(\alpha_0)] \times [0,1]$, we define

$$\theta = \alpha_0 - \varepsilon t,$$

$$\delta(\rho, t) = \frac{1}{\varepsilon}\left[L_{\beta_0}(\alpha_0 - \varepsilon t) - L_{\beta_0}(\alpha_0)\right] \chi\left(\frac{L_{\beta_0}(\alpha_0) - \rho}{L_{\beta_0}(\alpha_0)}\right),$$

$$r = \rho + \varepsilon\delta(r, t).$$

Observe that δ also depends on ε.

The mapping Φ is seen to be a diffeomorphism from $[0, L_{\beta_0}(\alpha_0)] \times [0,1]$ onto W_ε, which is a neighbourhood in T of the side $\{(r, \alpha_0) : r \in [0, L_{\beta_0}]\}$ that shrinks when ε goes to 0. Using Φ as a change of variables, we obtain

$$\mathbb{A}_\varepsilon(h) = \int_0^1 \int_0^{L_{\beta_0}(\alpha_0)} \chi'(t) h(\rho + \varepsilon\delta, \alpha_0 - \varepsilon t)\left[1 + \varepsilon\partial_\rho\delta\right](\rho + \varepsilon\delta) d\rho dt.$$

We obtain a sum of three terms:

$$\mathbb{A}_\varepsilon^0(h) = \int_0^1 \int_0^{L_{\beta_0}(\alpha_0)} \chi'(t) h(\rho + \varepsilon\delta, \alpha_0 - \varepsilon t)\rho d\rho dt,$$

$$R_\varepsilon^1(h) = \int_0^1 \int_0^{L_{\beta_0}(\alpha_0)} \chi'(t) h(\rho + \varepsilon\delta, \alpha_0 - \varepsilon t)\varepsilon\partial_\rho\delta\rho d\rho dt,$$

$$R_\varepsilon^2(h) = \int_0^1 \int_0^{L_{\beta_0}(\alpha_0)} \chi'(t) h(\rho + \varepsilon\delta, \alpha_0 - \varepsilon t)\left[1 + \varepsilon\partial_\rho\delta\right]\varepsilon\delta) d\rho dt.$$

Using that δ is identically 0 if $\rho < \frac{1}{3}$ and that δ and its first-order derivatives are uniformly bounded with respect to ε, t, ρ, by undoing the change of variables we see that the terms R^i, $i = 1, 2$ are bounded by the integral of $|h|$ over W_ε. Since $h \in L^1$ the latter goes to 0 and we are left to study the limit of $\mathbb{A}_\varepsilon^0$.

We now set

$$R_\varepsilon^0(h) = \mathbb{A}_\varepsilon^0(h) + \int_0^{L_{\beta_0}(\alpha_0)} h(r, \alpha_0) r\, dr$$

so that, as above

$$R_\varepsilon^0(h) = \int_0^1 \int_0^{L_{\beta_0}(\alpha_0)} \chi'(t)\left[h(\rho + \varepsilon\delta, \alpha_0 - \varepsilon t) - h(\rho, \alpha_0)\right] \rho d\rho dt.$$

Fix $(\rho, t) \in [0, L_{\beta_0}(\alpha_0)] \times [0,1]$. The mapping $\gamma(\cdot\,; \rho, t) : s \mapsto (\rho + s\varepsilon\delta(\rho, st), \alpha_0 - \varepsilon st)$ sends the interval $[0,1]$ onto a smooth curve in T that stays within W_ε. Since h is W_1^1, we have

$$R_\varepsilon^0(h) = \int_0^1 \int_0^{L_{\beta_0}(\alpha_0)} \int_0^1 \chi'(t)\big[\varepsilon(\delta(\rho, st) + st\partial_t\delta(\rho, st))\partial_r h \circ \gamma(s\,; \rho, t)$$

$$- \varepsilon t\partial_\theta \circ \gamma(s\,; \rho, t)\big]\rho ds d\rho dt.$$

We now argue in a similar fashion as before: we define Ψ on $[0, L_{\beta_0}(\alpha_0)] \times [0, 1] \times [0, 1]$ by $\Psi(\rho, t, s) = (\rho, \gamma(s \, ; \rho, t))$. We show that it is a legitimate change of variables, which turns R_ε^0 into an integral that is bounded above by the integral of a L^1 function over W_ε. It thus tends to 0 and that makes the proof finally complete.

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INSTITUT DENIS POISSON, UNIVERSITÉ DE TOURS ET UNIVERSITÉ D'ORLÉANS, PARC DE GRANDMONT, 37200 TOURS, FRANCE

Email address: `luc.hillairet@univ-orleans.fr`

CNRS AND INSTITUT DENIS POISSON, UNIVERSITÉ DE TOURS ET UNIVERSITÉ D'ORLÉANS, PARC DE GRANDMONT, 37200 TOURS, FRANCE

Email address: `helen.jenne@lmpt.univ-tours.fr`

CNRS AND LABORATOIRE ANGEVIN DE RECHERCHE EN MATHÉMATIQUES, UNIVERSITÉ D'ANGERS, 2 BOULEVARD LAVOISIER, 49000 ANGERS, FRANCE

Email address: `raschel@math.cnrs.fr`