Full asymptotic expansion for orbit-summable quadrant walks and discrete polyharmonic functions

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Abstract

Enumeration of walks with small steps in the quadrant has been a topic of great interest in combinatorics over the last few years. In this article, it is shown how to compute exact asymptotics of the number of such walks with fixed start- and endpoints for orbit-summable models with finite group, up to arbitrary precision. The resulting representation greatly resembles one conjectured for walks starting from the origin in [18], differing only in terms appearing due to the periodicity of the model. We will see that the dependency on start- and endpoint is given by discrete polyharmonic functions, which are solutions of $\triangle^n v = 0$ for a discretisation $\triangle$ of a Laplace-Beltrami operator. They can be decomposed into a sum of products of lower order polyharmonic functions of either the start- or the endpoint only, which leads to a partial extension of a theorem by Denisov and Wachtel in [22].

1 Introduction and Motivation

Enumeration of lattice paths has by now become a standard problem in combinatorics. In particular the asymptotics of lattice path counting problem in the half- and quarter plane have been addressed in a variety of works. To do so, the by now standard approach as used in [2] is to consider the generating function

$$Q(x, y; t) = \sum_{n \in \mathbb{N}} \sum_{k,l \in \mathbb{Z}} x^k y^l t^n q(0, B; n), \tag{1}$$

where $q(0, B; n)$ denotes the (possibly weighted) number of lattice paths starting at $(0, 0)$ and ending at a point $B = (k, l)$. While for walks in the entire plane this contains negative powers of r and s and is thus not a function in these two variables, it is one in t. Utilizing a functional equation for $Q(x, y; t)$, it is then often possible to extract information about the asymptotic behaviour of $q(0, B; n)$, as was done for instance in [2], where the authors gain various asymptotic expressions for the case of a directed model (i.e. the first component of a step is always positive) closely tied to the zeros of the so-called kernel. In the undirected case, restricting ourselves to walks in the quarter plane, asymptotics of the coefficients of $Q(1, 1; t)$ or $Q(0, 0; t)$ (that is, of the walks starting at the origin and ending either anywhere or back at the origin) have been computed for some cases in [27], using a complex boundary value problem, and for a family of models related to the Gouyou-Beauchamps model in [20] via Analytic Combinatorics in Several Variables (ACSV, [37]). In [9], the authors give hypergeometric expressions in the 23 cases where the group is finite (see [15]); and even explicit formulas in four cases where the model was known to be. To do so, they use orbit summation paired with a guessing approach. Additionally, they obtain the qualitative result that in the remaining 19 cases, the function $Q(1, 1; t)$ is transcendental in t. This goes in a similar direction as [8], and in general the question about the algebraic properties of the generating function $Q(x, y; t)$. This function is known to be D-finite in x, y, t if and only if the group is finite, which was first conjectured in [15] and subsequently proven by various authors, for instance [7, 40, 34, 31].

There have also been extensions of the usual setting, where either steps are not required to be small or not to be homogeneous [25], [33], or where the setting is not in the quarter plane but some other

domain, e.g. in 3 or more dimensions [10], [16], [39], [28], [3].

Instead of counting all walks of a certain length, however, one could also sort them by their endpoint. For some standard models, like for example the Simple Walk ($\mathcal{S} = \{\leftarrow, \uparrow, \rightarrow, \downarrow\}$), or the Tandem Walk ($\mathcal{S} = \{\leftarrow, \uparrow, \searrow\}$), explicit formulas for $q(0, B; n)$ are given in [15], making use of coefficient extraction and orbit summation. One of the main results in the aforementioned article is that one can make use of orbit-summation in order to obtain an expression for the generating function $Q(x, y; t)$ in order to show it is D-finite, but it turns out that this approach can be utilized to compute asymptotics of $q(0, B; n)$ to arbitrary high order instead. For highly symmetric models, this is used in [38] in order to find diagonal representation of the generating functions, and then use ACSV to compute the asymptotics of the coefficients of $Q(1, 1; t)$. In [39], this is then refined to all orbit-summable models with finite group. Seeing as the principal idea behind the saddle point method and ACSV is in fact rather similar (even though the latter is more general and uses much heavier machinery), it seems likely that their approach could be generalized to compute exact asymptotics of $q(k, l; n)$ as well. A general first order approximation with fixed start- and endpoint is given in [22], where using coupling with Brownian motion the authors show that, under some moment conditions, we have

$$q(A, B; n) \sim \frac{V(A)\tilde{V}(B)}{n^c} + o\left(n^{-c}\right), \tag{2}$$

where $q(A, B; n)$ is the number of paths with starting- and endpoints $A = (u, v)$ and $B = (k, l)$ respectively, and $V(A), \tilde{V}(B)$ are discrete harmonic functions in u, v and k, l respectively.

In [18], the authors used methods from [4] to show that for a Brownian motion, the asymptotic expansion of the heat kernel with respect to time contains (continuous) polyharmonic functions. They noticed as well that a discrete analogue of this exists for the Simple Walk, and conjectured that this might also be the case for other models. Using orbit summation together with a saddle point method, we will show this to be true in a more general context in Thm. 1 and Thm. 2. In particular, we will show that for all orbit-summable models (see Sec. 2), obtain for any given $m \in \mathbb{N}$ asymptotics of the form

$$q(A, B; n) = \frac{\gamma^n}{n^c}\left[\sum_{p=1}^{m-1} \frac{v_p(A, B)\sum_{i=1}^r \alpha_i^{u-k}\beta_i^{v-l}\zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right)\right], \tag{3}$$

where $c \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and the $\alpha_i, \beta_i, \zeta_i$ are roots of unity. The coefficients $v_p(x, y)$ are so-called discrete polyharmonic functions (for a formal definition see Sec. 2) of order p in B and, in reversed direction, in A. If the model has no drift then they are polynomials, otherwise they contain additional exponential factors. Compared to (3), the expansion appearing in [18] is different in the lack of the $\alpha_i, \beta_i, \zeta_i$, which appear due to periodicity properties: at certain points we might have $q(A, B; n) = 0$, and at those points some kind of cancellation needs to occur.

The structure of the polyharmonic functions $v_p(x, y)$ will be described more closely in Thm. 3, where it will be shown that we can write

$$v_p(A, B) = \sum_{i=1}^k h_{p,i}(A)g_{p,i}(B), \tag{4}$$

where the $h_i(A)$ and $g_i(B)$ are (in the case of y adjoint) polyharmonic of degree at most p in A and B respectively (this will be detailed in Thm. 3), and the number of summands k depends on p. For $p = 1$ this reduces precisely to (2), so in a certain sense one could view Thm. 3 as an extension (albeit under much stronger conditions) of ([22, Thm. 7]).

While the results of this article are stated for orbit-summable small step models in two dimensions, they do generalize to higher dimensions and – with a slight condition – to orbit-summable large step models as well, see the remarks after Thm. 1 as well as A and B.

This article will be structured as follows:

1. In Section 2, a short overview over some definitions and the tools utilized will be given.

2. In Section 3, it will be shown that models with finite group allowing for orbit summation in a manner similar as in [15] satisfy an asymptotic relation of the form (3), and the example of the

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Gouyou-Beauchamps model will be worked out in detail. In particular it will be explicitly shown how to compute the constants γ, c, $\alpha_i, \beta_i, \zeta_i$ and the functions $v_p(B)$ in (3).

3. In Section 4, we will consider the same problem where instead of $(0,0)$, we start our paths at an arbitrary point (u,v). It turns out, maybe not surprisingly, that due to the symmetry of the problem the resulting solution is similar to that obtained in Section 3, leading to Thm. 2. Continuing from there, we can then find a decomposition of the v_p as in (4) in Thm. 3. The resulting decomposition of the asymptotic terms for the Simple Walk is given in Section 4.2, using an explicit basis of polyharmonic functions constructed in [41].

4. In App. A and App. B, examples of the method applied to a model with large steps and a three-dimensional model are given.

5. In App. C, a decomposition of the polyharmonic coefficients as in Sec. 4.2 will be done for the Gouyou-Beauchamps model and the Tandem Walk.

6. Finally, in App. D, the first three terms of the asymptotics for all the 19 unweighted, orbit-summable models are given.

2 Preliminaries

2.1 Walks in the Quarter Plane

In order to count lattice paths in the quarter plane, first of all we need a set $\mathcal{S} \subseteq \mathbb{Z}^2$ of permissible steps, together with a family of weights $(\omega_s)_{s \in \mathcal{S}}$. A lattice path of length n is then a sequence of points $(x_0, \ldots, x_n) \subseteq \mathcal{Q}^n$, with $\mathcal{Q} := \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ being the quarter plane, such that $x_k - x_{k-1} \in \mathcal{S}$ for all $k \in \{1, \ldots, n\}$. We will count such a path by its weight, that is, the product $\prod_{k=1}^n \omega_{x_k - x_{k-1}}$.
We will in the following assume that:

1. our step set consists of small steps only, i.e. $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$,

2. our step set is non-degenerate, i.e. there is no (possibly rotated) half-plane containing all allowed steps.

In order to keep the notation short, we will in the following sometimes denote by $\mathcal{S}$ not only the set of allowed steps, but also the associated weights $(\omega_s)_{s \in \mathcal{S}}$.
Let $q((u,v), (k,l); n)$ be the (weighted) number of paths of length n from (u,v) to the endpoint (k,l). It can then be shown (see e.g. [15, Lemma 4]) that the generating function

$$Q_{u,v}(x, y; t) = \sum_{n \geq 0} t^n \sum_{k,l \in \mathbb{N}} x^k y^l q((u,v), (k,l); n) \tag{5}$$

satisfies the functional equation

$$K(x, y; t) Q_{u,v}(x, y; t)$$
$$= x^{u+1} y^{v+1} - \underbrace{K(x, 0; t) Q_{u,v}(x, 0; t)}_{=:A(x;t)} - \underbrace{K(0, y; t) Q_{u,v}(0, y; t)}_{=:B(y;t)} + \underbrace{K(0, 0; t) Q_{u,v}(0, 0; t)}_{=:C(t)}, \tag{6}$$

where

$$K(x, y) = xy \left[1 - t S(x, y)\right], \tag{7}$$

where $S(x, y)$ is the step-counting Laurent polynomial

$$S(x, y) = \sum_{(i,j) \in \mathcal{S}} \omega_{(i,j)} x^i y^j. \tag{8}$$

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For ease of notation, if $(u,v) = (0,0)$, then we will just write $q(k,l;n)$ and $Q(x,y;t)$.

Given that we consider paths in the quarter plane only, we see that the series $Q_{u,v}(x,y;t)$ are indeed power series in x,y and t; in particular they are power series in t with polynomial coefficients in x,y. Given some power series $A(x,y) = \sum_{i,j\in\mathbb{N}} a_{i,j}x^iy^j$, we denote by $[x^n]$ the linear operator extracting the n-th coefficient, i.e. $[x^n]A(x,y) = \sum_{j\in\mathbb{N}} a_{n,j}y^j$, and $[x^ny^m]A(x,y) = a_{n,m}$. In the same manner, we define for Laurent series the operator $[x^>] := \sum_{n>0}^{\infty} x^n[x^n]$ extracting all positive powers with their coefficients, and the operator $[x^{\geq}]$ extracting all nonnegative ones.

Seeing as we allow small steps only and our step set is non-degenerate, we can write the kernel $K(x,y)$ as

$$K(x,y) = a(x)y^2 + b(x)y + c(x) \tag{9}$$
$$= \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \tag{10}$$

with $a(x), b(x), c(x), \tilde{a}(y), \tilde{b}(y), \tilde{c}(y)$ all being non-zero. Consider now the two birational transformations

$$\Phi : \begin{cases} x \mapsto x, \\ y \mapsto y^{-1}\frac{c(x)}{a(x)}, \end{cases} \tag{11}$$

$$\Psi : \begin{cases} x \mapsto x^{-1}\frac{\tilde{c}(y)}{\tilde{a}(y)}, \\ y \mapsto y. \end{cases} \tag{12}$$

These two transformations, which clearly depend on our step set $\mathcal{S}$, generate the so-called **group of a model**, denoted by $\mathcal{G}$. This group can be either finite or infinite. If $\mathcal{G}$ is finite, then as both Φ, Ψ are involutions, any element $g \in \mathcal{G}$ can be written as either $(\Phi \circ \Psi)^k$, or as $\Psi \circ (\Phi \circ \Psi)^k$ for some k. We define $\text{sgn}(g) = 1$ in the first, and $\text{sgn}(g) = -1$ in the second case. The study of this group has been central to many results about lattice walks in the quarter plane, see e.g. [15, 31, 23, 27]. The main reason for this is that if we let

$$k(x,y) := \frac{K(x,y)}{xy} = 1 - tS(x,y), \tag{13}$$

then it follows that $k(x,y)$ is invariant under $\mathcal{G}$. This observation is the starting point for orbit summation methods as is done in [15]: we rewrite (6) (leaving out the dependency on t for easier readability) as

$$xyk(x,y)Q(x,y) = xy - A(x) - B(y) + C. \tag{14}$$

Assuming now that the group is finite, and picking any $g \in \mathcal{G}$, we hence obtain

$$g(xy)k(x,y)Q(g(x),g(y)) = g(xy) - A(g(x)) - B(g(y)) - C. \tag{15}$$

Multiplying (15) with $\text{sgn}(g)$ and taking the sum over all elements $g \in \mathcal{G}$, all terms $A(g(x))$ and $B(g(y))$ cancel (note that both Ψ, Φ change only one variable each, while sgn switches sign), we obtain (see [15, Prop. 5])

$$\sum_{g\in\mathcal{G}} \text{sgn}(g)g(xy)Q(g(x),g(y)) = \frac{1}{k(x,y)}\sum_{g\in\mathcal{G}} \text{sgn}(g)g(xy). \tag{16}$$

Depending on the exact shape of $\mathcal{G}$, often the terms of the form $Q(g(x),g(y))$ will not contribute positive powers of both x and y except for $g = \text{id}$. In this case, we say that the model is **orbit-summable**, and we can write

$$xyQ(x,y;t) = [x^>]\,[y^>]\,\frac{\sum_{g\in\mathcal{G}} \text{sgn}(g)g(xy)}{k(x,y;t)}. \tag{17}$$

By [15, Lemma 2], we know that if a model $\mathcal{S}$ (consisting of both steps and weights ω_s) has a finite group, then the model with reversed steps $\tilde{S} := -\mathcal{S}$ has a finite group as well. Furthermore, by essentially the same argument, one can see that orbit-summability is retained when reversing the steps as well:

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Lemma 1. *Given an orbit-summable model $\mathcal{S}$, the reversed model $\tilde{\mathcal{S}}$ is also orbit-summable.*

Proof. We note as in [15, Lemma 2] that if Φ, Ψ are the generators of the group of $\mathcal{S}$, then the generators of the group of $\tilde{\mathcal{S}}$ are given by $\iota \circ \Psi \circ \iota, \iota \circ \Phi \circ \iota$, where $\iota(x, y) = \left(x^{-1}, y^{-1}\right)$. Given any element g in the group $\mathcal{G}$ of $\mathcal{S}$, denote by $\tilde{g}$ the corresponding element of the group $\tilde{\mathcal{G}}$ of $\tilde{\mathcal{S}}$. As ι is an involution, we see that $\tilde{g} = \iota \circ g \circ \iota$ for any such g, and in particular that $\tilde{g}(xy) = \iota \circ g \circ \iota(xy)$. Noticing that $g(xy) \in \mathbb{C}(x, y)$ we can deduce that $\tilde{g}(xy) = g(xy)$. From there, we obtain an equivalent equation to (17) by the very same combinatorial arguments. □

In the following, it will be convenient to consider models with zero drift, that is, where

$$\sum_{(i,j)\in\mathcal{S}} i\omega_{i,j} = \sum_{(i,j)\in\mathcal{S}} j\omega_{i,j} = 0.$$

If this is not the case, then we can utilize the **Cramer-transformation**: we multiply each weight $\omega_{i,j}$ by a factor of $\alpha^i\beta^j$, where we choose α, β such that the drift will be 0. The existence of such an α, β is ensured for non-singular models, see e.g. [22, p. 1.5]. The reason why this substitution is very convenient combinatorially is fairly simple; given the number $q((i,j),(k,l);n)$ of paths from (i,j) to (k,l) with n steps weighted by old weights $\omega_{i,j}$, then for the equivalent $\hat{q}$ using the new weights we have

$$\hat{q}((i,j),(k,l);n) = \alpha^{k-i}\beta^{j-l}q((i,j),(k,l);n). \tag{18}$$

Additionally, it turns out that the group of the model is, in a certain sense, invariant under this transformation, and in particular orbit-summability is preserved.

Lemma 2. *Let $\mathcal{S}$ be a model and $\hat{\mathcal{S}}$ be a Cramer-transform of $\mathcal{S}$, with weights α, β. Let $\mathcal{G}, \hat{\mathcal{G}}$ be the respective groups with generators Φ, Ψ and $\hat{\Phi}, \hat{\Psi}$. Lastly, let ι be the mapping $(x, y) \mapsto (\alpha x, \beta y)$. Then we have:*

1. *$\hat{\Phi}(x, y) = \iota^{-1} \circ \Phi \circ \iota, \hat{\Psi}(x, y) = \iota^{-1} \circ \Psi \circ \iota$; in particular $\mathcal{G}$ and $\hat{\mathcal{G}}$ are isomorphic,*

2. *$\hat{\mathcal{S}}$ is orbit-summable if and only if $\mathcal{S}$ is orbit-summable.*

Proof. We show the first part for Φ only, the statement for Ψ will then follow by symmetry. We can define $\hat{c}(x)$ and $\hat{a}(x)$ as in (9). One finds that, by definition of $\hat{\mathcal{S}}$, we will have $\hat{a}(x) = \beta a(\alpha x)$ and $\hat{c}(x) = \beta^{-1}c(\alpha x)$. From this it follows that

$$\iota^{-1} \circ \Phi \circ \iota(x, y) = \iota^{-1} \circ \Phi(\alpha x, \beta y) = \iota^{-1}\left[\alpha x, (\beta y)^{-1}\frac{c(\alpha x)}{a(\alpha x)}\right] \tag{19}$$

$$= \iota^{-1}\left(\alpha x, \beta y^{-1}\frac{\hat{c}(x)}{\hat{a}(x)}\right) = \left(x, y^{-1}\frac{\hat{c}(x)}{\hat{a}(x)}\right) = \hat{\Phi}(x, y). \tag{20}$$

The isomorphism of $\mathcal{G}$ and $\hat{\mathcal{G}}$ follows immediately and is given by $g \mapsto \iota^{-1} \circ g \circ \iota \in \hat{\mathcal{G}}$. By this isomorphism, we also see immediately that positive and negative powers of x, y are preserved, which is in turn all that matters for orbit-summability, hence we are done. □

Given a model with zero drift, one can consider the correlation coefficient θ (see [27, 45]) given by

$$\theta = \arccos\left(-\frac{\sum_{(i,j)\in\mathcal{S}} ij\omega_{i,j}}{\sqrt{\sum_{(i,j)\in\mathcal{S}} i^2\omega_{i,j} \cdot \sum_{(i,j)\in\mathcal{S}} j^2\omega_{i,j}}}\right). \tag{21}$$

It turns out that this correlation coefficient is closely linked to the properties of the group (its restriction to the surface $\mathcal{C} := \{(x, y) : K(x, y) = 0\}$ is finite if and only if $\pi/\theta \in \mathbb{Q}$), and also has a geometric interpretation (see [45]). Additionally, in [43], [32], it is shown that π/θ gives the growth of the generating function of the positive harmonic function at the point $(1, 1)$. Given a model with non-zero drift, we can first apply a Cramer-transform to get rid of the drift, and then define θ as above.

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2.2 Discrete Polyharmonic Functions

Given a step set $\mathcal{S}$ and a discrete function f defined on the quarter plane $\mathcal{Q}$, we can define the Markov operator

$$Pf(x) := \sum_{s \in \mathcal{S}} \omega_s f(x - s). \tag{22}$$

If we take the discrete random walk (X_n) with the transition probabilities given by the reverse of $\mathcal{S}$, and the induced Markov chain $M_n := f(X_n)$, then we can interpret the operator P as the expectation $\mathbb{E}[M_{n+1} \mid M_n]$. One can then proceed to look at the expected change during a time step, weighted by a parameter t, which is given by

$$\triangle f(x) := (P - t\,\mathrm{id})\, f(x). \tag{23}$$

The operator $\triangle$ can be viewed as the discrete equivalent of a Laplace-Beltrami operator. We call a function **(discrete)** t**–harmonic** if

1. $\triangle f(x) = 0$ for all $x \in \mathcal{Q}$,

2. $f(x) = 0$ for all $x \in \mathcal{Q}^c$,

where $\mathcal{Q}^c$ is the complement of $\mathcal{Q}$. Similarly, we call a function **(discrete)** t**–polyharmonic of degree** p if

1. $\triangle^p f(x) = 0$ for all $x \in \mathcal{Q}$,

2. $f(x) = 0$ for all $x \in \mathcal{Q}^c$.

If $t = 1$, then we simply speak of harmonic and polyharmonic functions respectively. Note that strictly speaking, the polyharmonicity is given by the first conditions, while the second ones are Dirichlet boundary conditions. Due to the underlying combinatorics, however, one immediately sees that we are only interested in polyharmonic functions satisfying the Dirichlet problem as above. In particular (1-)harmonic and biharmonic functions have a variety of applications in the theory of stochastic processes and physics, see e.g. [35, 1, 19]. The occurrence of t-polyharmonic functions in the asymptotics of path counting problems, however, was noted only fairly recently in [18]. In [41], it is shown that in the zero drift case, the space of 1-polyharmonic functions of order n is isomorphic to $\mathbb{C}[[z]]^n$, and a basis consisting of polyharmonic functions with rational generating function is given.

Given a model $\mathcal{S}$, we can also define a discrete Laplacian $\tilde{\triangle}$ for the model with directions reversed $\tilde{\mathcal{S}}$. Seeing as $\tilde{\triangle}$ is the adjoint operator to $\triangle$ on the space $\mathcal{L}^2(\mathbb{Z}^2)$, we will call it the adjoint Laplacian. In Section 4.1, we will encounter functions of the form $f(x, y)$, for $x, y \in \mathbb{Z}^2$, which are polyharmonic in x and adjoint polyharmonic in y. We will call such a function $f(x, y)$ **multivariate polyharmonic** of order p if

$$\triangle^k \left(\tilde{\triangle}^{p-k} f \right) = 0$$

for all $0 \le k \le p$. Note that in this case, the ordering of the Laplacians does not matter as by linearity they commute, i.e. we have

$$\triangle \left(\tilde{\triangle} f(x, y) \right) = \tilde{\triangle} \left(\triangle f(x, y) \right). \tag{24}$$

Lastly, discrete polyharmonic functions behave well with respect to Cramer transformations: via a short computation, one can show

Lemma 3. *Let $\hat{\mathcal{S}}$ be a Cramer-transform of $\mathcal{S}$, with $\hat{\omega}_{i,j} = \alpha^i \beta^j \omega_{i,j}$. Let $\hat{\triangle}, \triangle$ be the associated Laplacians. We then have*

$$\hat{\triangle} \left[\alpha^{-k} \beta^{-l} f(k, l) \right] = \alpha^{-k} \beta^{-k} \triangle \left[f(k, l) \right]. \tag{25}$$

This directly implies that we have a bijection between the polyharmonic functions w.r.t. $\triangle$ and those w.r.t. $\hat{\triangle}$, given by adding a factor of $\alpha^{-k} \beta^{-l}$.

2.3 Saddle Points

In the following we will use the saddle point method as described for instance in [29, p. VIII]. In particular, we will be interested in saddle points of $S(x, y)$, which is a Laurent polynomial with only positive coefficients. We call a point (x_0, y_0) a **dominant saddle point** if:

1. (x_0, y_0) is a local minimizer of $S(x, y)$,

2. $x_0, y_0 \in \mathbb{R}^+$.

The fact that $S(x_0, y_0) \neq 0$ follows from the positivity of coefficients. The existence is a consequence of the fact that our model is non-degenerate; which implies that $S(x, y)$ is coercive (it goes to infinity wherever we approach the boundary; see also [22, p. 1.5]). By definition, we know that $\frac{\partial S}{\partial x}(x_0, y_0) = \frac{\partial S}{\partial y}(x_0, y_0) = 0$.
By a short computation, one can check that

Lemma 4. *The dominant saddle point $s_0 = (x_0, y_0)$ is equal to $(1, 1)$ if and only if the drift of the model is 0.*

Now take any other point (x_1, y_1) on the torus $|x_1| = x_0, |y_1| = y_0$. By the positivity of the coefficients of $S(x, y)$, we know that $|S(x_1, y_1)| \leq S(x_0, y_0)$, unless there is a $\zeta \in \mathbb{C}$, $|\zeta| = 1$, such that each monomial term of $S(x_1, y_1)$ differs from the corresponding term in $S(x_0, y_0)$ by the same factor of ζ. Choosing α, β such that $(x_1, y_1) = (\alpha x_0, \beta y_0)$, we must therefore have $\alpha^i \beta^j = \zeta$ for all (i, j) such that $\omega_{i,j} \neq 0$. From here, it is not difficult to see that α, β and ζ must be roots of unity. It is also clear that there can only be finitely many such ζ, in a one-to-one correspondence with finitely many pairs $(x_i, y_i), 0 \leq i \leq l$ (the maximum l appearing for the 19 unweighted orbit-summable models is 3, see App. D). We will call such a point $(x_i, y_i) = (\alpha_i x_0, \beta_i y_0)$ and $|S(x_i, y_i)| = S(x_0, y_0)$ a **saddle point associated with** (x_0, y_0). One can check directly that the (x_i, y_i) are indeed saddle points of $S(x, y)$ as well, moreover, as we will see in Lemma 5, the local behaviour of $S(x_0, y_0)$ and $S(x_i, y_i)$ is the same up to the factor ζ. By the same reasoning as in [29, p. VIII], it turns out that when computing the asymptotics of the coefficients via the Cauchy formula, the main contributions to the contour integral come from the points (x_i, y_i), as the modulus of $S(x, y)$ will be smaller elsewhere, leading to exponentially smaller terms.
Note that while this article considers mainly the 2-dimensional case, all the definitions above extend to more dimensions in a natural manner; only the structure of the group becomes more complicated as we will have more than two transformations, see e.g. [10].

3 Quadrant walks starting at the origin

3.1 Full asymptotic expansion

The goal of this section is to compute the asymptotics of orbit-summable lattice walks from the origin to an arbitrary but fixed point in the quarter plane, and in particular to show the following:

Theorem 1. *Let $\mathcal{S}$ be a step set satisfying the general assumptions stated in Sec. 2 such that $\mathcal{S}$ is orbit-summable, i.e.*

$$xyQ(x, y; t) = \left[x^> \right] \left[y^> \right] \frac{N(x, y)}{k(x, y; t)},$$

where

$$N(x, y) := \sum_{g \in \mathcal{G}} \operatorname{sgn}(g) g(xy).$$

Suppose that $s_0 = (x_0, y_0)$ is a dominant saddle point, with associated other saddle points s_i, $1 \leq i \leq r$ (meaning that we consider $r + 1$ saddle points in total). Furthermore, let $\alpha_i, \beta_i, \zeta_i$ be the roots of unity

as constructed in Sec. 2. Then, there are a constant $c \in \mathbb{N}$, a constant $\gamma > 0$, and γ-polyharmonic functions v_p of degree p such that for any $m \in \mathbb{N}$ we have

$$q(0, (k,l); n) = \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{m-1} \frac{v_p(k,l) \sum_{i=1}^r \alpha_i^k \beta_i^l \zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right) \right]. \qquad (26)$$

The polyharmonic functions $v_p(k,l)$ are polynomials precisely if the drift of the model is zero, else they contain an additional factor of $x_0^{-k} y_0^{-l}$, with (x_0, y_0) the dominant saddle point. They can be computed explicitly via a Cauchy-type integral.
Lastly, the constant c can be expressed using the correlation coefficient θ defined in (21) via $c = \pi/\theta$.

In order to keep the proof of Thm. 1 reasonably concise, we will first start with two somewhat technical lemmas. In the first lemma, we will establish some periodicity properties of $S(x, y)$ and the group, in the case where we have multiple saddle points.

Lemma 5. *Let (x_0, y_0) be a dominant saddle point, and $(x_i, y_i) = (\alpha_i x_0, \beta_i x_0)$ be the associated ones, with $S(x_i, y_i) = \zeta_i S(x_0, y_0)$. Let furthermore $\Phi(x, y) = (x, \phi(x, y))$, $\Psi(x, y) = (\psi(x, y), y)$ be the generators of the group as in (11),(12). We then have, for all $x, y \in \mathbb{C}$:*

$$S(\alpha_i x, \beta_i y) = \zeta_i S(x, y), \qquad (27)$$
$$\psi(\alpha_i x, \beta_i y) = \alpha_i \psi(x, y), \qquad (28)$$
$$\phi(\alpha_i x, \beta_i y) = \beta_i \phi(x, y). \qquad (29)$$

Remark: Lemma 5 still holds true in more than two dimensions, with a completely analogous proof.

Proof. As (x_0, y_0) is a dominant saddle point and (x_i, y_i) associated to it, we have $|\alpha_i| = |\beta_i| = |\zeta_i| = 1$, and know that for each monomial $x^k y^l$ appearing in $S(x, y)$ we have $(\alpha_i x)^k (\beta_i y)^l = \zeta_i x^k y^l$. Consequently, for all such k, l we have $\alpha_i^k \beta_i^l = \zeta_i$, and thus (27) holds.
We can define $\psi(x, y)$ almost everywhere by the properties

$$S(\psi(x, y), y) = S(x, y), \qquad (30)$$
$$\psi(x, y) \neq x, \qquad (31)$$

the exception being the points where $\frac{\partial S}{\partial x}(\cdot, y) = 0$, i.e. where x is uniquely defined by the property $S(\cdot, y) = S(x, y)$. We know that for any y, we have $\alpha_i x \neq \alpha_i \psi(x, y)$ almost everywhere. Furthermore, using (27), we can rewrite:

$$S(\alpha_i \psi(x, y), \beta_i y) = \zeta_i S(\psi(x, y), y) = \zeta_i S(x, y) = S(\alpha_i x, \beta_i y). \qquad (32)$$

Note in particular that (27) and therefore (32) hold true for all $x, y \in \mathbb{C}$, not only saddle points. Since ψ is rational, we therefore have (28), and by symmetry also (29). $\qquad \square$

Lastly, we will show that given an asymptotic representation as in Thm. 1 below, the functions appearing therein are indeed polyharmonic.

Lemma 6. *Suppose $q(B; n)$ is a (combinatorial) quantity satisfying*

$$q(B; n+1) = \sum_{x \in \mathcal{S}} \omega_s q(x - s; n) \quad \forall B \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, n \geq 0, \qquad (33)$$

and at the same time it is of the form

$$q(B; n) = \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^r v_{p,i}(B) \zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^{m+1}}\right) \right], \qquad (34)$$

for all $k \geq 0$, with the ζ_i pairwise different roots of unity. Then, the $v_{p,i}$ are γ-polyharmonic of degree p.
If, additionally, for a fixed point B we know that $q(B; n) = 0 \ \forall n \in \mathbb{N}$, then $v_{p,i}(B) = 0$ for all p, i.

Proof. Substituting (34) into (33) gives us

$$q(B; n+1) = \sum_{s \in \mathcal{S}} \omega_s q(B-s; n) \quad \Leftrightarrow$$

$$\gamma^{n+1} \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^{r} v_{p,i}(B) \zeta_i^{n+1}}{(n+1)^{p+c}} + \mathcal{O}\left(\frac{1}{n^{m+c}}\right) \right] = \gamma^n \sum_{s \in \mathcal{S}} \omega_s \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^{r} v_{p,i}(B-s) \zeta_i^n}{n^{p+c}} \right].$$

Extracting the terms for $p = 1$ and noticing that the others are smaller by a factor of at least $\frac{1}{n}$, we obtain

$$\gamma \frac{n^{c+1}}{(n+1)^{c+1}} \sum_{i=1}^{r} v_{1,i}(B) \zeta_i^{n+1} + \mathcal{O}\left(\frac{1}{n}\right) = \sum_{s \in \mathcal{S}} \omega_s \sum_{i=1}^{r} v_{1,i}(B-s) \zeta_i^n. \tag{35}$$

Letting n go to ∞, we notice that the fraction $\frac{n^{c+1}}{(n+1)^{c+1}}$ will converge to 1. This gives us

$$\gamma \sum_{i=1}^{r} v_{1,i}(B) \zeta_i^{n+1} + \mathcal{O}\left(\frac{1}{n}\right) = \sum_{s \in \mathcal{S}} \omega_s \sum_{i=1}^{r} v_{1,i}(B-s) \zeta_i^n. \tag{36}$$

All we need to do now is show that each of the $v_{1,i}(x)$ is γ-harmonic by itself, i.e. that (36) holds for each summation index separately. To do so, let us forget for a moment the part $\mathcal{O}\left(\frac{1}{n}\right)$ and solve the exact analogue to (36).
Since by assumption the ζ_i are all different, we know that the vectors

$$(\zeta_1^m, \ldots, \zeta_l^m), \quad 0 \le m \le r-1 \tag{37}$$

are linearly independent (written as a matrix, they give a Vandermonde matrix with determinant $\prod_{i<j}(\zeta_j - \zeta_i) \ne 0$). Therefore, the system

$$\sum_{i=1}^{r} \left(c v_{i,1}(B) - \sum_{s \in \mathcal{S}} \omega_s v_{i,1}(B-s) \right) \zeta_i^n = 0 \tag{38}$$

has no nontrivial solutions, and hence

$$c v_{i,1}(B) = \sum_{s \in \mathcal{S}} \omega_s v_{i,1}(B-s) \quad \forall 1 \le i \le l. \tag{39}$$

All that remains to do now is to see that the error term $\mathcal{O}\left(\frac{1}{n}\right)$ in (36) does not change anything. To do so, suppose now that (39) is not satisfied for some i. Then we know that there are arbitrarily large n such that (38) does not hold, i.e. its right-hand side takes a value $\varepsilon_n \ne 0$. As the ζ_i are roots of unity, there are only finitely many values which ε_n can take for different n, so we cannot have convergence of ε_n to 0. But then, choosing n large enough, (35) cannot hold either; a contradiction. Thus, (39) must hold, and we know that the $v_{1,i}(B)$ are harmonic. By induction, applying the discrete Laplacian $\triangle$ to both sides of (35), we argue in the same fashion that the $v_{k,i}(B)$ must be polyharmonic of degree k. For the second part, suppose that $q(B; n) = 0$ for all n, but we have p, i such that $v_{p,i}(B) \ne 0$. Assume our p to be minimal with this property. Then, we know that

$$\sum_{i=1}^{l} v_{p,i}(B) \zeta_i^n = 0, \tag{40}$$

because otherwise this would be a contradiction to (34) for large n. But then we can utilize independence of the vectors $(\zeta_1^k, \zeta_2^k, \ldots, \zeta_l^k)$ for $k = 0, \ldots, l-1$ as before and arrive at a contradiction. Finally, to show that the $v_{p,i}$ are polyharmonic of order p, we can apply the operator $\triangle^{p-1}$ to both sides of (34), notice that the first $p-1$ terms vanish by assumption and then repeat the above argument. $\quad \square$

Remarks:

- In the proof of Thm. 1 we will see that in our case, for different i the $v_{p,i}(k,l)$ differ only by a factor of $\alpha_i^{-k}\beta_i^{-l}$. Here, given a dominant saddle point (x_0, y_0), and an associated one (x_i, y_i), then α_i, β_i are the numbers such that $(x_i, y_i) = (\alpha_i x_0, \beta_i y_0)$. This allows us to essentially talk about only a single polyharmonic function v_p for any given p; namely the one defined by the dominant saddle point.

- We will see in Section 4.1 that we always have $\triangle v_{p+1} = v_p + r_{p-1}$, where r_{p-1} is some polyharmonic function of degree at most $p-1$. This hints at the fact that the polyharmonic functions appearing in the asymptotic expansions are not arbitrary, but do have some form of recursive structure.

We now have all ingredients ready for the proof of Thm. 1.

Proof (of Thm. 1). By the assumptions, we know that we have

$$q(0, (k,l); n) = \left[x^k y^l t^n\right] Q(x, y; t) = \left[x^{k+1} y^{l+1} t^n\right] \frac{\sum_{g \in \mathcal{G}} \operatorname{sgn}(g) g(xy)}{k(x, y; t)}. \tag{41}$$

As $k(x, y) = 1 - tS(x, y)$, we can rewrite

$$q(0, (k,l); n) = \left[x^{k+1} y^{l+1} t^n\right] \sum_{i=0}^{\infty} t^i S(x, y)^i N(x, y) \tag{42}$$

$$= \left[x^{k+1} y^{l+1}\right] S(x, y)^n N(x, y). \tag{43}$$

By Cauchy's formula, we have

$$q(0, (k,l); n) = -\frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{S(x, y)^n N(x, y)}{x^{k+2} y^{l+2}} \mathrm{d}x\mathrm{d}y, \tag{44}$$

with $\Gamma_{1,2}$ being closed curves around the origin. To evaluate the asymptotics of this integral, we utilize the saddle point method, as described for instance in [29, Chapter VIII].

The main idea is to conveniently choose our contours Γ_1, Γ_2 such that they make the integral as easy to compute as possible.

With this in mind, suppose that (x_0, y_0) is a dominant saddle point, and pick $\Gamma_1 = \{|x| = |x_0|\}, \Gamma_2 = \{|y| = |y_0|\}$. We know that the modulus of $S(x, y)$ on $\Gamma_1 \cap \Gamma_2$ is maximal; and the only other points where it attains the same value are the associated saddle points (x_i, y_i). At any other point, $|S(x, y)|$ will be strictly smaller – hence, when n goes to infinity, it suffices to compute the integral locally around our saddle points, since the rest will grow exponentially slower. We could hypothetically run into issues if $N(x, y)$ were to be infinite, but we will see that this is not the case at our saddle points (and for any other point, we can always just slightly shift our contour to avoid a pole). It is easy to check that, given an $\varepsilon > 0$, the set $|S(x_0, y_0) - S(x, y)| < \varepsilon$ is contained in a domain of the form $|x - x_0| < \varepsilon_1, |y - y_0| < \varepsilon_2$. As previously mentioned, the rest of the integral can be – in order to find the asymptotics – neglected, as it will be exponentially smaller. Changing our coordinates to $x = x_0 e^{is/\sqrt{n}}, y = y_0 e^{it/\sqrt{n}}$, this corresponds to a region of the form $\left|\frac{s}{\sqrt{n}}\right| < \delta_1, \left|\frac{t}{\sqrt{n}}\right| < \delta_2$, or, equivalently, $|s| < \delta_1 \sqrt{n}, |t| < \delta_2 \sqrt{n}$.

To find the asymptotics, it therefore suffices to compute the integrals

$$\int_{-\delta_1 \sqrt{n}}^{\delta_1 \sqrt{n}} \int_{-\delta_2 \sqrt{n}}^{\delta_2 \sqrt{n}} F_j(s, t, k, l, n) \mathrm{d}t\mathrm{d}s, \tag{45}$$

where $F_j(s, t, k, l, n)$ is the expression obtained by substituting $x = x_j e^{is/\sqrt{n}}, y = y_j e^{it/\sqrt{n}}$, for (x_j, y_j) the relevant saddle points (i.e. (x_0, y_0) and the ones associated to it, as outlined in Section 2). We will see that, given any fixed $m \in \mathbb{N}$, each such integral can be written in the form

$$\frac{\gamma^n}{n^c} \int_{-\delta_1 \sqrt{n}}^{\delta_1 \sqrt{n}} \int_{-\delta_2 \sqrt{n}}^{\delta_2 \sqrt{n}} e^{-Q(s,t)} \left(p_0(s, t) + \frac{p_1(s, t)}{n} + \frac{p_2(s, t)}{n^2} + \cdots + \mathcal{O}\left(\frac{1}{n^m}\right) \right) \mathrm{d}t\mathrm{d}s, \tag{46}$$

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where $Q(s,t)$ is some (positive definite) quadratic form and the $p_j(s,t)$ are polynomials. Assume w.l.o.g. that $\delta_1 \geq \delta_2$. For $s, t > \delta_2 \sqrt{n}$, we can see that the integral over the remaining part of $\mathbb{R}$ is exponentially small in n, and therefore we can consider the integral

$$\frac{\gamma^n}{n^c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-Q(s,t)} \left(p_0(s,t) + \frac{p_1(s,t)}{n} + \cdots + \mathcal{O}\left(\frac{1}{n^m}\right) \right) \mathrm{d}t\mathrm{d}s. \tag{47}$$

Consequently, all we need to do is to consider the latter integral for all saddle points with maximum absolute value (of which there are finitely many), and by computing all the expressions up to a fixed $p_j(s,t)$, we will then have obtained the asymptotics of (44) and at the same time shown (26).

In the following, we will proceed in two steps: first, we pick a dominant saddle point and show that, locally around this point, everything works out smoothly. Then, we pick any associated saddle point and show that, up to powers of roots of unity, nothing changes from the first step.

1. Suppose (x_0, y_0) is a dominant saddle point, let $\gamma := S(x_0, y_0)$ and fix a $m \in \mathbb{N}$. First, we show that $0 = |N(x_0, y_0)| < \infty$. This is due to (x_0, y_0) being a saddle point: we have $\frac{\partial S}{\partial y}(x_0, y_0) = 0$ and thus y_0 is the unique solution to $S(x_0, \cdot) = S(x_0, y_0)$. Therefore, $\phi(y_0) = y_0$, and in the same manner we can see $\psi(x_0) = x_0$. It follows immediately that the alternating orbit sum of xy evaluated at (x_0, y_0) is 0; in particular it is finite.

   Our next step is to show that we can rewrite the integrand (that is, the one in (44)) as in (47). To do so, we substitute $x \mapsto x_0 \exp \frac{is}{\sqrt{n}} =: e_s, y \mapsto y_0 \exp \frac{it}{\sqrt{n}} =: e_t$, and then separate the integral into three parts: $S(e_s, e_t)^n$, $N(e_s, e_t)$ and the denominator $1/e_s^{k+1} e_t^{k+1}$ (note that a power in the denominator vanishes due to the substitution rule).

   (a) Part 1: $S(e_s, e_t)^n$
       As (x_0, y_0) is a dominant saddle point of $S(x, y)$ and therefore $\frac{\partial S}{\partial x}(x_0, y_0) = \frac{\partial S}{\partial y}(x_0, y_0) = 0$, we have a Taylor expansion of $S(x, y)$ around (x_0, y_0) of the form

       $$S(x, y) = \gamma - u(x - x_0)^2 - v(x - x_0)(y - y_0) - w(y - y_0)^2 + \ldots \tag{48}$$

       After our substitution, this gives us (note that e_s, e_t are functions of s, t and n)

       $$S(e_s, e_t) = \gamma - u\frac{s^2}{4n} - v\frac{st}{n} - w\frac{t^2}{4n} + A(s, t, n), \tag{49}$$

       with $A(s, t, n) = n^{-3/2} \sum_{j=0}^{2m-1} \frac{a_j(s,t)}{n^{j/2}} + \mathcal{O}\left(n^{-m-3/2}\right)$, and the r_j homogeneous polynomials. We know that $\widehat{Q}(s, t) := \left[\frac{u}{4}s^2 + vst + \frac{w}{4}t^2\right]$ is a positive definite quadratic form (seeing as our saddle point is a local minimizer of $S(x, y)$ in $\mathbb{R}^+ \times \mathbb{R}^+$). Consequently, we can write

       $$\log S(e_s, e_t) = \log\left(\gamma - \left[\widehat{Q}(s, t) - A(s, t, n)\right]\right) \tag{50}$$

       $$= \log \gamma - \underbrace{\frac{1}{\gamma}\widehat{Q}(s, t)}_{=:Q(s,t)} + B(s, t, n), \tag{51}$$

       with once again $B(s, t, n) = n^{-3/2} \sum_{j=0}^{2m-1} \frac{b_j(s,t)}{n^{j/2}} + \mathcal{O}\left(n^{-m-3/2}\right)$, the b_j' homogeneous polynomials and $Q(s, t)$ a positive definite quadratic form. Consequently,

       $$S(e_s, e_t)^n = \exp\left[n \log S(e_s, e_t)\right] = \gamma^n \exp\left[-Q(s, t)\right] \exp\left[nB(s, t, n)\right]. \tag{52}$$

       Comparing this to (44), the first two factors are already precisely as we want them, and the last factor is of the form $\sum_{m \geq 1} \frac{q_m(s,t)}{n^{m/2}}$, with the $q_m(s, t)$ homogeneous of degree m.

   (b) Part 2: $N(e_s, e_t)$
       As we have seen that $|N(x_0, y_0)| < \infty$, and seeing as $N(x, y)$ is a rational function in x, y, it follows that we can write

       $$N(e_s, e_t) = \sum_{j=0}^{2m-1} \frac{d_j(s,t)}{n^{j/2}} + \mathcal{O}\left(n^{-m-1}\right), \tag{53}$$

       with the $d_j(s, t)$ homogeneous polynomials of degree j.

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(c) Part 3: $1/e_s^{k+1}e_t^{l+1}$
Lastly, we have

$$\frac{1}{e_s^{k+1}e_t^{l+1}} = x_0^{-k-1}y_0^{-l-1}\sum_{j\geq 0}\frac{1}{n^{j/2}}\frac{(-i[(k+1)s+(l+1)t])^j}{j!}. \tag{54}$$

Note in particular that this last factor is the only part which depends on the endpoint (k,l).

Multiplying the power series together and sorting them by powers of n, we obtain as a result the contribution of this saddle point to (47) of the form

$$\frac{\gamma^n}{n^c}\left[\sum_{p=0}^{m-1}\frac{1}{n^{p/2}}\frac{1}{x_0^{k+1}y_0^{l+1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q(s,t)}q_p'(s,t,k,l)\mathrm{d}s\mathrm{d}t + \mathcal{O}\left(\frac{1}{n^{m-1/2}}\right)\right], \tag{55}$$

where the $q_{i,p}'(s,t,k,l)$ are polynomials in s,t,k,l. One can easily see, however, that $q_p'(s,t,k,l)$ is homogeneous of degree p; for odd p the double integral therefore vanishes by symmetry. Thus we can rewrite (55) as

$$\frac{\gamma}{n^c}\left[\sum_{p=0}^{m-1}\frac{1}{n^p}\frac{1}{x_0^{k+1}y_0^{l+1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q(s,t)}q_p(s,t,k,l)\mathrm{d}s\mathrm{d}t + \mathcal{O}\left(\frac{1}{n^m}\right)\right], \tag{56}$$

with $q_p := q_{2p}'$. The factor $\frac{1}{n^c}$ in (55) stems from the fact that we obtain a factor of $1/n$ by the substitution rule, and that the integral might vanish for small values of p.

2. Suppose now that we have another saddle point $(x_i, y_i) = (\alpha_i x_0, \beta_i y_0)$ associated to (x_0, y_0), and pick ζ_i such that $S(x_i, y_i) = \zeta_i S(x_0, y_0)$ (notice that we then have $|\zeta_i| = 1$, as discussed in Section 2). Our goal is now to describe the series expansion of the numerator around (x_i, y_i) using the one around (x_0, y_0). We substitute as before $x \mapsto e_s' := x_i\exp\frac{is}{\sqrt{n}}, y \mapsto e_t' := y_i\exp\frac{it}{\sqrt{n}}$. Due to Lemma 5, we can now conclude that the series representation (w.r.t. n at ∞) around $S(e_s', e_t')$ is the same as the one of $\zeta_i S(e_s, e_t)$, and the representation of $N(e_s', e_t')$ is the same as $\alpha_i\beta_i N(e_s, e_t)$. Lastly, the expansion of $1/e_s'^{(k+1)}e_t'^{(l+1)}$ clearly changes only by adding a factor of $\alpha_i^{k+1}\beta_i^{l+1}$ as well. Therefore, we can conclude that the contribution of the saddle point (x_i, y_i) is the same as the one of (x_0, y_0) up to a factor of $\zeta_i^n\alpha_i^{-k}\beta_i^{-k}$.

By Lemma 6, we deduce that the $v_p(k,l)$ are indeed γ-polyharmonic of degree p (note that for our dominant saddle point we have $\alpha = \beta = \zeta = 1$).

Seeing as the only point in the construction where k,l appear is in (54), one finds that the $v_p(k,l)$ are, up to a factor of $x_0^{-k}y_0^{-l}$, bivariate polynomials in k,l. Due to Lemma 4, they are therefore polynomials if and only if the model has zero drift.

The fact that $c = \pi/\theta$ (with θ the correlation coefficient as defined in (21)) follows from [22, Thm. 7]. $\square$

The construction used in the proof allows us to give some further properties of the polyharmonic functions appearing in the asymptotic expansion.

Corollary 1. *The degree of the polynomial part of $v_p(k,l)$ (that is, without the factor of $x_0^{-k}y_0^{-l}$) is $c + 2p - 1$.*

Proof. By looking once again at the proof of Thm. 1 and in particular (54). From there, the statement follows immediately. $\square$

Corollary 2. *For any orbit-summable model, we have $\pi/\theta \in \mathbb{N}$.*

Remarks:

- Thm. 1 holds true in higher dimensions as well, and indeed the proof translates directly. The one difference lies in the powers of n which appear: by the substitution rule $\frac{\mathrm{d}x_i}{\mathrm{d}s_i} = c_i e^{is_i/\sqrt{n}}$, one obtains an additional factor of $n^{-1/2}$. Thus, for even dimensions the constant c in (26) will be integer, whereas for odd dimensions it will be in $\frac{1}{2} + \mathbb{N}$. An example case for three dimensions is treated in App. B.

- When looking at models with large steps, the one thing that could go wrong is that the numerator might have a singularity at a saddle point. Usually, this seems not to be the case, and for a given model this condition is very easy to check. An example is treated in App. A.

- A table of the first three asymptotic terms for the 19 unweighted orbit-summable models is given in App. D.

3.2 Example: the Gouyou-Beauchamps model

In this section, we will illustrate the result of Thm. 1 by computing the asymptotics for the Gouyou-Beauchamps model, and in doing so find an explicit formula for the polyharmonic functions appearing therein.

The Gouyou-Beauchamps Walk is defined by the step set $\{\nwarrow, \searrow, \leftarrow, \rightarrow\}$. Its step polynomial is $S(x, y) = \frac{y}{x} + \frac{x}{y} + \frac{1}{x} + x$, and solving $S_x = S_y = 0$ yields the four solutions $(x, y) = (\pm 1, \pm 1)$. We find that $S(1, 1) = 4, S(-1, 1) = -4$, $S(1, -1) = S(-1, -1) = 0$. Verifying that the second derivatives do not vanish, we therefore have the dominant saddle point $(1, 1)$ and one other associated to it, namely $(-1, 1)$, to consider. Note that the appearance of two saddle points is not at all surprising here, due to parity (or, in more general terms, periodicity) considerations: depending on the first coordinate, we can only hit a point after an even or odd number of steps. Therefore we can already expect at this point the asymptotics resulting from the saddle points to be precisely the same up to a factor of $(-1)^{k+n}$, which coincides with the statement of Thm. 1. Hence, we will only consider the dominant saddle point here.

The alternating orbit sum of xy can be checked to be

$$N(x, y) := -\frac{(-1 + x^2)(-1 + y)(x^4 + y^3 - x^2 y - x^2 y^2)}{x^3 y^2}. \tag{57}$$

Out integrand is therefore of the form

$$\frac{S(x, y)^n N(x, y)}{x^{k+2} y^{l+2}},$$

where after letting $x \mapsto e_s := e^{is/\sqrt{n}}, y \mapsto e_t := e^{it/\sqrt{n}}$, by the substitution rule we will end up with

$$-\frac{1}{n} \frac{S(e_s, e_t)^n N(e_s, e_t)}{e_s^{k+1} e_t^{l+1}}.$$

The first factor of $-1/n$ is entirely harmless; we will therefore proceed to compute each of the factors in the second fraction separately.

1. Series representation of $S(e_s, e_t)^n$:
   First, we can compute (we let $m := \sqrt{n}$ for readability)

   $$S(e_s, e_t) = 4 - \sum_{j \geq 2} \frac{a_j}{m^j} = 4 - \underbrace{\frac{s^2 + (s - t)^2}{n}}_{=:4Q(s,t)} + \underbrace{\sum_{j \geq 3} \frac{a_j}{m^j}}_{=:A(s,t,m)}, \tag{58}$$

   with

   $$a_j := i^j \frac{s^j + (s - t)^j}{j!} (1 + (-1)^j).$$

   From here we obtain

   $$\log [S(e_s, e_t)] = \log 4 - \frac{Q(s, t)}{n} + B(s, t, m), \tag{59}$$

where $B(s,t,m) = \sum_{j \geq 3} \frac{b_j}{m^j}$, with

$$b_j = \sum_{i_1 + i_2 + \cdots + i_n = j} \frac{(-1/4)^n}{n} \prod_{p=1}^{n} a_{i_p},$$

with the i_j positive integers.

Finally, we can compute

$$S(e_s, e_t)^n = \exp\left(n \log\left[S(e_s, e_t)\right]\right) \tag{60}$$
$$= \exp\left(n \log\left[4 + B(s,t,m)\right] - Q(s,t)\right) \tag{61}$$
$$= 4^n e^{-Q(s,t)} \underbrace{\exp\left[B(s,t,m)\right]}_{:=C(s,t,m)}, \tag{62}$$

where $C(s,t,m) = \sum_{k \geq 1} \frac{c_j}{j^k}$ with

$$c_j = \sum_{i_1 + \cdots + i_n = j} \frac{1}{n!} \prod_{p=1}^{n} b_{i_p},$$

with the i_j again positive integers.

2. Series representation of $N(e_s, e_t)$:
   Using (57), we find that

$$N(e_s, e_t) = \sum_{j \geq 1} \frac{d_j}{m^j}, \tag{63}$$

with

$$d_j := \frac{i^j}{j!}\left[-(-4)^j s^j + (2s - 3t)^j - 2^j(s-t)^j - (t-2s)^j + (t-4s)^j + (-2)^j(s+t)^j - (-3)^j t^j\right].$$

3. Series representation of $1/e_s^{k+1} e_t^{l+1}$:
   Lastly, we have

$$\frac{1}{e_s^{k+1} e_t^{l+1}} = \exp\left[-i(s(k+1) + t(l+1))/m\right] = \sum_{j \geq 0} \frac{1}{m^j} \underbrace{\frac{(-i((k+1)s + (l+1)t))^j}{j!}}_{=:f_j}.$$

Overall, we obtain as product of the three factors computed above

$$4^n e^{-Q(s,t)} \sum_{p \geq 0} \frac{1}{m^p} \underbrace{\left[\sum_{j_1 + j_2 + j_3 = p} c_{j_1} d_{j_2} f_{j_3}\right]}_{=:q_p},$$

for j_1, j_2, j_3 nonnegative integers. In particular, we notice that q_p is homogeneous of degree p in s, t, and of degree p in k, l. In order to compute the contribution up to order $\mathcal{O}\left(\frac{1}{n^j}\right)$ of this saddle point to the asymptotics, all we need to do now is compute

$$4^n \frac{1}{4\pi^2 n} \sum_{p=0}^{2r-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-Q(s,t)} q_p(s,t,k,l) dt ds.$$

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This gives us an explicit formula for the asymptotics of this model. In particular, we can check directly that all coefficients of $\frac{1}{m^k}$ for odd k vanish. By computing the integrals, we see that

$$
\begin{aligned}
v_1(k,l) =& \frac{64}{\pi}(1+k)(1+l)(2+k+l)(3+k+2l), \\
v_2(k,l) =& -\frac{32}{\pi}(1+k)(1+l)(2+k+l)(3+k+2l)(35+2k^2+4k(2+l)+4l(3+l)), \\
v_3(k,l) =& \frac{8}{\pi}(1+k)(1+l)(2+k+l)(3+k+2l)(25+2k^2+4k(2+l) \\
& +4l(3+l))(61+2k^2+4k(2+l)+4l(3+l)).
\end{aligned}
$$

The harmonic function $v_1(k,l)$ has already been computed some time ago in [5, 44].

3.3 Periodicity

Considering the combinatorial context, it is clear that in any asymptotic expansion as in (26), the discrete harmonic function $v_1(k,l)$ will always be positive. When looking at the computations in App. D, it appears as if there is an even stronger pattern; namely that $v_p(k,l)$ is positive for even p, and negative for odd p. It turns out, however, that this is not generally true; a counterexample is given for instance by computing enough terms in the expansion of the Simple Walk [17].

It is also a direct consequence of Thm. 1 that the number of saddle points is closely tied to the periodicity of the model. If we have a single saddle point, then clearly our model is aperiodic; but the number of saddle points also corresponds directly to the periodicity.

Lemma 7. *Suppose that our model is irreducible (that is, the step set $\mathcal{S}$ generates all of $\mathbb{Z}^2$), and that it is m-periodic. Then we have exactly $m-1$ saddle points $s_1, \ldots, s_{m-1}$ associated to our dominant saddle point $s_0 = (x_0, y_0)$. The corresponding ζ_i are – in some order – the m-th roots of unity.*

Proof. Let r be the number of saddle points. We use the representation (26) for the asymptotics of $q(k,l;n)$. We know that the $\alpha_i, \beta_i, \zeta_i$ are roots of unity. We can therefore pick k,l such that $\alpha_i^k = \beta_i^l = 1$ for all i. We then know, since our model is m-periodic, that there is a z, $0 \le z \le m-1$, such that we have

$$
\chi(n) := 1 + \zeta_1^n + \cdots + \zeta_{r-1}^n = \begin{cases} 0 & n \not\equiv z \mod m, \\ m & n \equiv z \mod m. \end{cases} \tag{64}
$$

Note that as value for $\chi(k \cdot m), k \in \mathbb{Z}$, we could pick any constant, because to compensate we can just multiply the corresponding polyharmonic functions in (26) with a constant factor. By definition, we know that χ is a character on $\mathbb{Z}/m\mathbb{Z}$. Therefore, it can be uniquely written as sum of irreducible characters [36], which in this case are all the m-th roots of unity. From this, and the fact that the ζ_i are pairwise different, it follows already that the ζ_i are (in some ordering) the m-th roots of unity. $\square$

The regularity condition in Lemma 7 is necessary, as can be seen for the Diagonal Walk for instance. This walk is 2-periodic, but we still have 4 saddle points. The reason for the two extra saddle points is that there are some points the walk will never reach, which, heuristically speaking, translates to two additional conditions on k,l for which two saddle points then do not suffice to express them.

4 Quadrant walks with arbitrary starting point

4.1 Full asymptotic expansion

If we want to count walks starting from a point other than $(0,0)$, then the only thing that changes is that we have a different monomial $x^{k+1}y^{l+1}$ in the functional equation (6). It follows that we can proceed in exactly the same manner as before in order to obtain an expression as in (17), where only the sum on the right hand side changes. Hence, we can proceed in a the same manner as for Thm. 1, which allows us to recover the result of [22, Thm. 7], which states that the first order term in the

asymptotics of the number of walks terminating at y and starting at x will be given – up to, once again, an exponential term and some power of n, and possible parity constraints – by the product of two functions; a harmonic function in x and a function in y which is adjoint harmonic. This is fairly natural, seeing as the underlying combinatorial problem is highly symmetrical: any path from x to y corresponds to a path from y to x with reversed steps. This, and a similar statement for the higher order terms, can be formalized in Thm. 2 below and the following Thm. 3.

Theorem 2. *Suppose that $\mathcal{S}$ is a step set satisfying the general assumptions stated in Section 2 and that $\mathcal{S}$ is orbit-summable. Then, with (x_0, y_0) being a dominant saddle point and (s_i) the associated ones such that $s_i = (x_i, y_i) = (\alpha_i x_0, \beta_i y_0)$ and $S(x_i, y_i) = \zeta_i S(x_0, y_0)$, then there are a constant $c \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$ as well as functions $v_p(k, l, u, v)$ such that for any $m \in \mathbb{N}$ we have*

$$q((u,v),(k,l);n) = \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{m-1} \frac{v_p(k,l,u,v) \sum_{i=1}^{r} \alpha_i^{u-k} \beta_i^{v-l} \zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right) \right]. \tag{65}$$

The $v_p(k, l, u, v)$ are polynomials precisely if the drift is zero (else they contain exponential factors). In this case they are of bidegree $c + 2p - 1$ in both (k, l) and (u, v), and of total degree $2c + 4p - 2$. Each $v_p(k, l, u, v)$ is multivariate polyharmonic of degree p.

Before proving Thm. 2, we first show the following lemma, which is a natural extension of Lemma 6.

Lemma 8. *Suppose that $q(A, B; n)$ is a (combinatorial) quantity satisfying*

$$\sum_{s \in \mathcal{S}} \omega_s q(A - s, B; n - 1) = q(A, B; n), \tag{66}$$

$$\sum_{s \in \mathcal{S}} \omega_s q(A, B + s; n - 1) = q(A, B; n), \tag{67}$$

for all $A, B \in \mathbb{Z}_{\geq} \times \mathbb{Z}_{\geq 0}$, $n \geq 0$. Assume furthermore that $q(A, B; n)$ has an asymptotic representation of the form

$$q(A, B; n) = \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{m-1} \frac{\sum_{i=1}^{l} v_{p,i}(A, B) \zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^m}\right) \right] \tag{68}$$

for all k, with the ζ_i pairwise different and of modulus 1. Then, each $v_{p,i}$ is multivariate polyharmonic of order p.

Proof. The proof works in the very same manner as the proof of Lemma 6; at each step we can choose whether to apply the identity (66) or (67), leading to an additional instance of $\triangle$ or $\tilde{\triangle}$ respectively. □

Proof (of Thm. 2). Analogous to the proof of Thm. 1. We note that the contribution of $N(e'_s, e'_y)$ for an associated saddle point changes by a factor of $\alpha_i^{u+1} \beta_i^{v+1}$ instead of a factor of $\alpha_i \beta_i$, and only need to keep track of the coefficients u, v throughout, which however behave exactly in the same manner as k, l.

Finally, the polyharmonicity properties are a direct consequence of Lemma 8. □

Remark:

- While the starting point (u, v) and the end point (k, l) of the walk end up playing a very similar role (see also Thm. 3), which is not at all surprising from a combinatorial point of view, this is not at all obvious from the proof: the role of (k, l) is very easily summarized as these coefficients only appear in the integrand as a factor of $x^{-k-1} y^{-l-1}$, the starting point does not appear directly as factor $x^{u+1} y^{v+1}$, but instead as its orbit sum. A priori, without the combinatorial interpretation, it does not seem to be obvious that both of these occurrences lead to a symmetrical role in the result.

In the following, we will want to describe the coefficients $v_p(k, l, u, v)$ appearing in Thm. 2 more precisely. We will consider the drift zero case, which is however not a real restriction as we can transform any other model via a zero drift one using the Cramer transformation discussed in Section 2. The goal will be to prove the following theorem:

Theorem 3. *The polynomials $v_p(k, l, u, v)$ in the zero drift case of Thm. 2 each have a representation of the form*

$$v_p(k, l, u, v) = \sum_{\substack{1 \leq i, j \leq p, \\ i+j \leq p+1}} a_{i,j} h_i^j(k, l) g_i^j(u, v), \tag{69}$$

where the $a_{i,j}$ are constants, the h_i^j are polyharmonic of degree (at most) i, and the g_i^j are adjoint polyharmonic of degree (at most) $p + 1 - i$.

In order to prove Thm. 3, it turns out to be very useful to have a polynomial basis of the space of polyharmonic functions for any given model. As the group is finite by assumption and $\pi/\theta \in \mathbb{Z}$ by Cor. 2, we can use the basis given in [41], which consists of sequences h_n^m of n-polyharmonic functions satisfying:

1. $\triangle h_{n+1}^m = h_n^m$,

2. the $h_n^m(k, l)$ are bivariate polynomials of increasing degree in both n and m: the degree will increase by 2 for each step in n, whereas it will increase by at least 2 and at most $c + 1$ for each step in m.

Proof (of Thm. 3). Taking u, v as parameters, we can for each (u, v) write $v_p(k, l, u, v)$ as a sum of the basis functions $h_i^j(k, l)$, and obtain

$$\sum_{\substack{1 \leq i, j \leq p, \\ i+j \leq p+1}} h_i^j(k, l) g_i^j(u, v). \tag{70}$$

Since we know that $v_p(k, l, u, v)$ is a bivariate polynomial of bidegree $c + 2p - 1$ in both (k, l) and (u, v) (which is also where the conditions $i, j \leq p$ and $i + j \leq p + 1$ come from), the only thing we need to show is that g_i^j is adjoint polyharmonic of degree $p + 1 - i$ for any j. To do so, we utilize Lemma 8. First, consider $i = p$. We then have

$$\tilde{\triangle} \left(\triangle^{p-1} q_p(k, l, u, v) \right) = \tilde{\triangle} h_1^1(k, l) g_p^1(u, v) = h_1^1(k, l) \tilde{\triangle} g_p^1(u, v) = 0, \tag{71}$$

therefore $v_p^1(u, v)$ is adjoint harmonic. Seeing as the discrete Laplacians are linear, we can now proceed by induction, in each step applying the same argument as above to all terms which are polyharmonic (in (k, l)) of order j, i.e. the multiples of $h_j^1, \ldots, h_j^{p+1-j}$. The statement follows. $\square$

Remarks:

- Thm. 3 tells us in particular that we can write each coefficient v_p as a product of polyharmonic and adjoint polyharmonic functions, so that the degree of the former and latter adds up to at most $p + 1$. This can be viewed as an extension of [22, Thm. 7], where an equivalent statement is shown for v_1 (albeit in a much more general setting).

- By simple degree considerations, the only base function h_p^j appearing in $v_p(k, l, u, v)$ can be h_p^1. From this it follows that $\triangle v_{p+1}(k, l, u, v) = v_p(k, l, u, v) + r_{p-1}(k, l, u, v)$, with r_{p-1} polyharmonic of degree at most $p - 1$.

- If the model is symmetric, then one can easily see that $a_{i,j} = a_{j,i}$ due to the symmetry of the underlying combinatorial problem ($q(A, B; n) = \tilde{q}(B, A; n)$, where $\tilde{q}$ denotes the paths with reversed steps). This holds true for an appropriate choice of basis in some other cases as well; for examples of this, see App. C.

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- A priori it is not at all clear which elements of the basis h_i^j constructed in [41] actually appear in connection with some combinatorial problems, i.e. if this basis is combinatorially reasonably chosen. As we will see in Sec. 4.2 and App. D, this seems to be the case.

- We can in fact give an upper bound on the number of summands appearing in the decomposition (69). First of all, we know for all p that all (i, j) such that h_i^j appears in the decomposition are contained in the subset $\{(i, j) : i + j \leq p + 1\}$. If one writes the h_i^j as a table, this gives us a triangular shape of size increasing with p. Along the lines $i + j = k$, we will have functions of degree at least $2(k - 1)$. We can now count the number of possible products of a given function h_i^j with a base function $\tilde{h}_m^n$ of the adjoint Laplacian. In order not to exceed the maximum degree, we can multiply h_1^1 with the entire triangle of adjoint base functions. For h_1^2 and h_2^1, we cannot multiply them with any $\tilde{h}_m^n$ with $m + n = p + 1$, and so on. All in all, this gives us a maximum of $\frac{p(p+1)(p+2)(p+3)}{24}$ summands. This maximum is achieved e.g. for the Simple Walk for $v_{1,2,3}$. Generally, the larger the value of π/θ, the less summands we will have (since the degrees of the h_i^j increase more quickly).

In the following, we will see what this decomposition looks like in case of the Simple Walk. The Gouyou-Beauchamps model and the Tandem Walk are treated in App. C.

4.2 Example: the Simple Walk

In the following, to keep the expressions a bit shorter, we will give the expressions as after the substitution $k \mapsto k - 1, l \mapsto l - 1$ etc. (i.e. we have kl instead of $(k + 1)(l + 1)$). This corresponds to a shift of the quarter plane, where instead of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ we now consider $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. For the Simple Walk, with $\mathcal{S} = \{\rightarrow, \downarrow, \leftarrow, \uparrow\}$, we then have (after rescaling by multiplicative constants)

$$
\begin{aligned}
h_1^1(k, l) &= kl, \\
h_1^2(k, l) &= kl(k - l)(k + l), \\
h_1^3(k, l) &= kl(14 - 5k^2 + 3k^4 - 5l^2 - 10k^2l^2 + 3l^4), \\
h_2^1(k, l) &= kl(l - 1)(l + 1), \\
h_2^2(k, l) &= kl(l - 1)(l + 1)(7 + 5k^2 - 3l^2), \\
h_3^1(k, l) &= kl(l - 2)(l - 1)(l + 1)(l + 2).
\end{aligned}
$$

By symmetry, we can pick the base functions $\tilde{h}_i^j(u, v) = h_i^j(u, v)$ for the adjoint Laplacian. For the first three asymptotic terms with arbitrary starting and ending points, we obtain (again up to multiplicative constants)

$$
\begin{aligned}
v_1(k, l, u, v) &= kluv, \\
v_2(k, l, u, v) &= kluv(7 + 2k^2 + 2l^2 + 2u^2 + 2v^2), \\
v_3(k, l, u, v) &= kluv(167 + 140k^2 + 12k^4 + 140l^2 + 24k^2l^2 + 12l^4 + 140u^2 + 40k^2u^2 \\
&\quad + 24l^2u^2 + 12u^4 + 140v^2 + 24k^2v^2 + 40l^2v^2 + 24u^2v^2 + 12v^4).
\end{aligned}
$$

One can check that Cor. 3 takes the form

$$
\begin{aligned}
v_1 &= h_1^1 \tilde{h}_1^1, \\
v_2 &= 4 \left(h_2^1 \tilde{h}_1^1 + h_1^1 \tilde{h}_2^1 \right) + 2 \left(h_1^2 \tilde{h}_1^1 + \tilde{h}_1^2 h_1^1 \right) + 15 h_1^1 \tilde{h}_1^1, \\
v_3 &= \frac{192}{5} \left(h_3^1 \tilde{h}_1^1 + \tilde{h}_3^1 h_1^1 \right) + \frac{64}{5} \left(h_2^2 \tilde{h}_1^1 + \tilde{h}_2^2 h_1^1 \right) + 4 \left(h_1^3 \tilde{h}_1^1 + \tilde{h}_1^3 h_1^1 \right) + 64 \left(h_2^1 \tilde{h}_1^2 + \tilde{h}_2^1 h_1^2 \right) \\
&\quad + 128 h_2^1 \tilde{h}_2^1 + 24 h_1^2 \tilde{h}_1^2 + 576 \left(h_2^1 \tilde{h}_1^1 + \tilde{h}_2^1 h_1^1 \right) + 288 \left(h_1^2 \tilde{h}_1^1 + \tilde{h}_1^2 h_1^1 \right) + 951 h_1^1 \tilde{h}_1^1.
\end{aligned}
$$

As the degree of h_i^j is truly increasing by only 2 whenever we increase either i or j by one, it turns out that we have indeed $1, 5$ and 15 different summands respectively. Due to the symmetry of this model,

the second and third equations can be simplified a bit: for v_2, letting $g_2 := 4h_2^1 + 2h_1^2$ (clearly, g_2 is then biharmonic) gives us

$$v_2 = g_2\tilde{h}_1^1 + h_1^1\tilde{g}_2 + 15h_1^1\tilde{h}_1^1.$$

For v_3, letting in the same manner $g_3 := \frac{192}{5}h_3^1 + \frac{64}{5}h_2^2 + 4h_1^3$, we have

$$g_3\tilde{h}_1^1 + \tilde{g}_3h_1^1 + 144\left(g_2\tilde{h}_1^1 + \tilde{g}_2h_1^1\right) + 64\left(h_2^1\tilde{h}_1^2 + \tilde{h}_2^1h_1^2\right) + 951h_1^1\tilde{h}_1^1.$$

While one can view the definition of $g_{2,3}$ as purely a crutch to make the resulting expressions shorter, they do in fact give in a sense a natural decomposition of the v_i: g_2 consists of the highest order terms in v_2, while g_3 consists of those in v_3.

Remarks:

- By the above, when taking a scaling limit, then we have

$$\lim_{\mu \to 0} m^\alpha v_i\left(\frac{x}{m}, \frac{y}{m}\right) = \lim_{\mu \to 0} m^\alpha g_i\left(\frac{x}{m}, \frac{y}{m}\right),$$

  where α is an appropriate scaling constant, which means that the g_i already give us all the terms which will not vanish in this kind of limit.

- By [18, Thm. 2.3], we know that the continuous heat kernel $p_t(x, y, u, v)$ of a Brownian motion with covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

  with $\sigma_{11} = \mathbb{E}[X^2], \sigma_{12} = \mathbb{E}[XY], \sigma_{22} = \mathbb{E}[Y^2]$ (the scaling limit of this model) allows for an asymptotic representation of the form

$$p_t(x, y, u, v) = \frac{1}{t^2}\sum_{k \geq 1} \frac{f_k(x, y, u, v)}{t^k}. \tag{72}$$

  Seeing as this representation looks almost the same as the one for the discrete case in Thm. 2, it is natural to compare the functions v_p to their continuous counterparts f_p. For v_1, for instance, by [22] we know that we will have (after appropriate scaling by a constant) $v_1(k, l, u, v) \to f_1(k, l, u, v)$. However, this is not at all clear for $p > 1$. For the Simple Walk, one can check that $v_2 \to f_2$, but this fails for $p = 3$: we have

$$f_3(k, l, u, v)$$
$$= kluv(3k^4 + 6k^2l^2 + 3l^4 + 22k^2u^2 + 6l^2u^2 + 3u^4 + 6k^2v^2 + 22l^2v^2 + 6u^2v^2 + 3v^4),$$

  whereas the scaling limit of $v_3(k, l, u, v)$ turns out to be

$$kluv(3k^4 + 6k^2l^2 + 3l^4 + 10k^2u^2 + 6l^2u^2 + 3u^4 + 6k^2v^2 + 10l^2v^2 + 6u^2v^2 + 3v^4),$$

  where the coefficients of k^2u^2 and l^2v^2 do not match.

5 Outlook

- We know by [22, Thm. 7] that, for any (not necessary orbit-summable) model with a finite number of steps, the first order term of the asymptotics of $q(k, l; n)$ behaves as in Thm. 1. However, not much is known about the higher order terms. In this context, one could view Thm. 1 as a partial generalization for the case of orbit-summable models with finite group, but it would be interesting to see whether something similar holds true for a more general class of models, which could for example also include models with large steps, as in e.g. [26]. While it appears at first that this might fail due to the more complicated structure of the group, in [6] the notion of an orbit is extended to a more general context, which might make an approach using a saddle point method feasible.

- While in the zero-drift case there is a unique combinatorially relevant harmonic function for a given model (namely the positive one), there is no equivalent in the polyharmonic case, nor a reasonable combinatorial interpretation. It would be very interesting to know if there is such an interpretation, or at least, in some sense, a 'canonical' p-polyharmonic function. Intuitively, one may think such a canonical function should coincide in its scaling limit with the (continuous) polyharmonic function appearing in the corresponding continuous heat kernel, but seeing as the higher order asymptotics of the discrete and continuous cases do not coincide (see Sec. 4.2), this might not be an ideal choice. Maybe the functions $g_2, g_3, \ldots$ as defined at the end of Section 4.2 could serve as candidates for such canonical representatives instead.

- While the saddle point method as used in this article works well to compute precise asymptotics for any given starting point given that the model has finite group and is orbit-summable, it seems that it would be hard to apply when one of these conditions does not hold. If the path counting function is algebraic (which implies a finite group), sometimes it is possible to obtain an explicit expression which can then be utilized to extract asymptotics [14, 11, 30]. In the infinite group case, however, things seem to be more complicated. In the one-dimensional setting, a probabilistic approach seems to work in order to show an asymptotic expansion similar to (3), using only moment conditions [21]. In two dimensions, one can tackle this problem using an explicit parametrization of the zero-set of the kernel via elliptic (in particular Jacobi theta-functions as in [42]). This method can then be utilized to show for some cases with infinite group (for instance the model with steps $\{\leftarrow, \uparrow, \rightarrow, \downarrow, \nearrow\}$) that the asymptotics of the number of walks returning to the origin contain logarithmic terms [24], which makes an asymptotic expansion as in (26) impossible. However, it turns out that the coefficients still have a similar structure in the sense that they consist of polyharmonic functions.

- Finally, one could pose similar questions for different domains, be it a higher dimension or a different cone, for instance the three quarter plane. See for example [13, 47, 46, 6, 12, 48, 10].

Acknowledgements

References

[1] Nakhlé H. Asmar and Loukas Grafakos. "Harmonic Functions and Applications". In: *Complex Analysis with Applications*. Cham: Springer International Publishing, 2018, pp. 367–402. ISBN: 978-3-319-94063-2. DOI: 10.1007/978-3-319-94063-2_6.

[2] Cyril Banderier and Philippe Flajolet. "Basic analytic combinatorics of directed lattice paths". In: *Theor. Comput. Sci.* 281.1-2 (2002), pp. 37–80. ISSN: 0304-3975. DOI: 10.1016/S0304-3975(02)00007-5.

[3] Cyril Banderier and Michael Wallner. "The kernel method for lattice paths below a line of rational slope". In: *Lattice path combinatorics and applications. Based on the 8th international conference on lattice path combinatorics and applications, California State Polytechnic University, Pomona (Cal Poly Pomona), CA, USA, August 17–20, 2015*. Cham: Springer, 2019, pp. 119–154. ISBN: 978-3-030-11101-4. DOI: 10.1007/978-3-030-11102-1_7.

[4] Rodrigo Bañuelos and Robert G. Smits. "Brownian motion in cones". In: *Probab. Theory Relat. Fields* 108.3 (1997), pp. 299–319. ISSN: 0178-8051. DOI: 10.1007/s004400050111.

[5] Philippe Biane. "Minuscule weights and random walks on lattices". In: *Quantum probability and related topics*. Singapore: World Scientific, 1992, pp. 51–65. ISBN: 981-02-1011-6.

[6] Alin Bostan, Mireille Bousquet-Mélou, and Stephen Melczer. "Counting walks with large steps in an orthant". In: *J. Eur. Math. Soc. (JEMS)* 23.7 (2021), pp. 2221–2297. ISSN: 1435-9855. DOI: 10.4171/JEMS/1053.

[7] Alin Bostan, Manuel Kauers, and Mark Van Hoeij. "The complete generating function for Gessel walks is algebraic". In: *Proc. Am. Math. Soc.* 138.9 (2010), pp. 3063–3078. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-2010-10398-2.

[8] Alin Bostan, Kilian Raschel, and Bruno Salvy. "Non-D-finite excursions in the quarter plane". In: *J. Comb. Theory, Ser. A* 121 (2014), pp. 45–63. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2013.09.005.

[9] Alin Bostan et al. "Hypergeometric expressions for generating functions of walks with small steps in the quarter plane". In: *Eur. J. Comb.* 61 (2017), pp. 242–275. ISSN: 0195-6698. DOI: doi:10.1016/j.ejc.2016.10.010.

[10] Alin Bostan et al. "On 3-dimensional lattice walks confined to the positive octant". In: *Ann. Comb.* 20.4 (2016), pp. 661–704. ISSN: 0218-0006. DOI: 10.1007/s00026-016-0328-7.

[11] Mireille Bousquet-Mélou. "An elementary solution of Gessel's walks in the quadrant". In: *Adv. Math.* 303 (2016), pp. 1171–1189. ISSN: 0001-8708. DOI: 10.1016/j.aim.2016.08.038.

[12] Mireille Bousquet-Mélou. *Enumeration of three-quadrant walks via invariants: some diagonally symmetric models.* 2021. arXiv: 2112.05776 [math.CO].

[13] Mireille Bousquet-Mélou. "Square lattice walks avoiding a quadrant". In: *J. Comb. Theory, Ser. A* 144 (2016), pp. 37–79. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2016.06.010.

[14] Mireille Bousquet-Mélou. "Walks in the quarter plane: Kreweras' algebraic model". English. In: *Ann. Appl. Probab.* 15.2 (2005), pp. 1451–1491. ISSN: 1050-5164. DOI: 10.1214/105051605000000052.

[15] Mireille Bousquet-Mélou and Marni Mishna. "Walks with small steps in the quarter plane". In: *Algorithmic probability and combinatorics. Papers from the AMS special sessions, Chicago, IL, USA, October 5–6, 2007 and Vancouver, BC, Canada, October 4–5, 2008.* Providence, RI: American Mathematical Society (AMS), 2010, pp. 1–39. ISBN: 978-0-8218-4783-1.

[16] Manfred Buchacher, Sophie Hofmanninger, and Manuel Kauers. "Walks with small steps in the 4D-orthant". In: *Ann. Comb.* 25.1 (2021), pp. 153–166. ISSN: 0218-0006. DOI: 10.1007/s00026-020-00520-5.

[17] François Chapon. Private correspondence.

[18] François Chapon, Éric Fusy, and Kilian Raschel. "Polyharmonic functions and random processes in cones". In: *31st international conference on probabilistic, combinatorial and asymptotic methods for the analysis of algorithms, AofA 2020, Klagenfurt, Austria (virtual conference), June 15–19, 2020.* Id/No 9. Wadern: Schloss Dagstuhl – Leibniz Zentrum für Informatik, 2020, p. 19. ISBN: 978-3-95977-147-4. DOI: 10.4230/LIPIcs.AofA.2020.9.

[19] Christopher I. Connoll and Roderic A. Grupen. "The applications of harmonic functions to robotics". In: *Journal of Field Robotics* 10 (7 1993), pp. 931–946. ISSN: 0097-3165. DOI: https://doi.org/10.1002/rob.4620100704.

[20] J. Courtiel et al. "Weighted lattice walks and universality classes". In: *J. Comb. Theory, Ser. A* 152 (2017), pp. 255–302. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2017.06.008.

[21] D. Denisov, A. Tarasov, and V. Wachtel. *Asymptotic expansions for first-passage times of a random walk.* In preparation. 2023.

[22] Denis Denisov and Vitali Wachtel. "Random walks in cones". In: *Ann. Probab.* 43.3 (2015), pp. 992–1044. ISSN: 0091-1798. DOI: 10.1214/13-AOP867.

[23] Thomas Dreyfus et al. "Walks in the quarter plane: genus zero case". In: *J. Comb. Theory, Ser. A* 174 (2020). Id/No 105251, p. 24. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2020.105251.

[24] Andrew Elvey-Price, Andreas Nessmann, and Kilian Raschel. *Logarithmic terms in discrete heat kernel expansions in the quadrant.* In preparation. 2023.

[25] Guy Fayolle and Kilian Raschel. "About a possible analytic approach for walks in the quarter plane with arbitrary big jumps". In: *C. R., Math., Acad. Sci. Paris* 353.2 (2015), pp. 89–94. ISSN: 1631-073X. DOI: 10.1016/j.crma.2014.11.015.

[26] Guy Fayolle and Kilian Raschel. "About a possible analytic approach for walks in the quarter plane with arbitrary big jumps". In: *C. R., Math., Acad. Sci. Paris* 353.2 (2015), pp. 89–94. ISSN: 1631-073X. DOI: 10.1016/j.crma.2014.11.015. URL: hal.inria.fr/hal-01021327/file/1-s2.0-S1631073X1400

[27] Guy Fayolle and Kilian Raschel. "Some exact asymptotics in the counting of walks in the quarter plane". In: *Proceeding of the 23rd international meeting on probabilistic, combinatorial, and asymptotic methods in the analysis of algorithms (AofA'12), Montreal, Canada, June 18–22, 2012*. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2012, pp. 109–124.

[28] Thomas Feierl. "Asymptotics for the number of walks in a Weyl chamber of type B". In: *Random Struct. Algorithms* 45.2 (2014), pp. 261–305. ISSN: 1042-9832. DOI: 10.1002/rsa.20467.

[29] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009. ISBN: 978-0-521-89806-5.

[30] L. Flatto and S. Hahn. "Two parallel queues created by arrivals with two demands. I". In: *SIAM J. Appl. Math.* 44 (1984), pp. 1041–1053. ISSN: 0036-1399. DOI: 10.1137/0144074.

[31] Charlotte Hardouin and Michael F. Singer. "On differentially algebraic generating series for walks in the quarter plane". In: *Sel. Math., New Ser.* 27.5 (2021). Id/No 89, p. 49. ISSN: 1022-1824. DOI: 10.1007/s00029-021-00703-9.

[32] Viet Hung Hoang, Kilian Raschel, and Pierre Tarrago. "Constructing discrete harmonic functions in wedges". In: *Trans. Am. Math. Soc.* 375.7 (2022), pp. 4741–4782. ISSN: 0002-9947. DOI: 10.1090/tran/8615.

[33] Manuel Kauers and Rika Yatchak. "Walks in the quarter plane with multiple steps". In: *Proceedings of the 27th international conference on formal power series and algebraic combinatorics, FPSAC 2015, Daejeon, South Korea, July 6–10, 2015*. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2015, pp. 25–36.

[34] Irina Kurkova and Kilian Raschel. "On the functions counting walks with small steps in the quarter plane". In: *Publ. Math., Inst. Hautes Étud. Sci.* 116 (2012), pp. 69–114. ISSN: 0073-8301. DOI: 10.1007/s10240-012-0045-7.

[35] S. A. Lurie and V. V. Vasiliev. *The Biharmonic Problem in the Theory of Elasticity*. Taylor & Francis Ltd, 1995. ISBN: 978-2884490542.

[36] H. Maschke. "Ueber den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionsgruppen." German. In: *Math. Ann.* 50 (1898), pp. 492–498. ISSN: 0025-5831. DOI: 10.1007/BF01444297.

[37] S. Melczer. *An Invitation to Analytic Combinatorics*. Springer International Publishing AG, 2021. ISBN: 978-3-030-67080-1.

[38] Stephen Melczer and Marni Mishna. "Asymptotic lattice path enumeration using diagonals". In: *Algorithmica* 75.4 (2016), pp. 782–811. ISSN: 0178-4617. DOI: 10.1007/s00453-015-0063-1.

[39] Stephen Melczer and Mark C. Wilson. "Higher dimensional lattice walks: connecting combinatorial and analytic behavior". In: *SIAM J. Discrete Math.* 33.4 (2019), pp. 2140–2174. ISSN: 0895-4801. DOI: 10.1137/18M1220856.

[40] Marni Mishna and Andrew Rechnitzer. "Two non-holonomic lattice walks in the quarter plane". In: *Theor. Comput. Sci.* 410.38-40 (2009), pp. 3616–3630. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2009.04.008.

[41] Andreas Nessmann. *Polyharmonic Functions in the Quarter Plane*. 2022. arXiv: 2212.07258 [math.CO].

[42] Andrew Elvey Price. "Counting lattice walks by winding angle". In: *Sémin. Lothar. Comb.* 84B (2020). Id/No 43, p. 12. ISSN: 1286-4889. URL: www.mat.univie.ac.at/~slc/wpapers/FPSAC2020//43.html.

[43] Kilian Raschel. "Counting walks in a quadrant: a unified approach via boundary value problems". In: *J. Eur. Math. Soc. (JEMS)* 14.3 (2012), pp. 749–777. ISSN: 1435-9855. DOI: 10.4171/JEMS/317.

[44] Kilian Raschel. "Green functions for killed random walks in the Weyl chamber of Sp(4)". In: *Ann. Inst. Henri Poincaré, Probab. Stat.* 47.4 (2011), pp. 1001–1019. ISSN: 0246-0203. DOI: 10.1214/10-AIHP405.

[45] Kilian Raschel. "Random walks in the quarter plane, discrete harmonic functions and conformal mappings". In: *Stochastic Processes Appl.* 124.10 (2014), pp. 3147–3178. ISSN: 0304-4149. DOI: 10.1016/j.spa.2014.04.013.

[46] Kilian Raschel and Amélie Trotignon. "On walks avoiding a quadrant". In: *Electron. J. Comb.* 26.3 (2019), research paper p3.31, 34. ISSN: 1077-8926. URL: www.combinatorics.org/ojs/index.php/eljc/article/vi

[47] Amélie Trotignon. "Discrete harmonic functions in the three-quarter plane". In: *Potential Anal.* 56.2 (2022), pp. 267–296. ISSN: 0926-2601. DOI: 10.1007/s11118-020-09884-y.

[48] Rika Yatchak. "Automated positive part extraction for lattice path generating functions in the octant". In: *Extended abstracts of the ninth European conference on combinatorics, graph theory and applications, EuroComb 2017, Vienna, Austria, August 28 – September 1, 2017*. Amsterdam: Elsevier, 2017, pp. 1061–1067. DOI: 10.1016/j.endm.2017.07.073.

A An example with large steps

Consider the model with steps $(1, 0), (-1, 0), (0, -1), (-2, 1)$. In [6, Prop. 16] it is shown that

$$Q(x, y; t) = \left[x^> y^>\right] \frac{(x^2 + 1)(x + y)(y - x)(x^2 y - 2x - y)(x^3 - x - 2y)}{x^7 y^3 (1 - tS(x, y))},$$

with $S(x, y)$ the step counting polynomial

$$S(x, y) = x + x^{-1} + y^{-1} + x^{-2} y.$$

We find the dominant saddle point to be at $s_0 := \left(\sqrt{3}, \sqrt{3}\right)$, and one other saddle point at $s_1 = \left(-\sqrt{3}, -\sqrt{3}\right)$ associated to it. We can check that we have[1], using the notation as in the proof of Thm. 1, $N(s_0) = 0$ (in particular it is not infinite), thus we can proceed in the same manner as in aforementioned theorem. We obtain

$$\gamma = 2\sqrt{3}, \quad c = 3,$$

and have

$$v_1(k, l) = \frac{16}{\pi} \cdot \sqrt{3}^{-1-k-l} (1 + k)(1 + l)(3 + k + 2l),$$

$$v_2(k, l) = -\frac{2}{\pi} \sqrt{3}^{-1-k-l} (1 + k)(1 + l)(3 + k + 2l)(107 + 4k^2 + 32l + 16l^2 + 8k(1 + l)),$$

$$v_3(k, l) = \frac{1}{8\pi} \sqrt{3}^{-3-k-l} (1 + k)(1 + l)(3 + k + 2l)(15205 + 16k^4 + 8672l + 4976l^2 + 832l^3 + 256l^4 + 64k^3(1 + l)$$
$$+ 8k^2(157 + 48l + 24l^2) + 16k(149 + 157l + 36l^2 + 16l^3)).$$

B A three-dimensional example

Consider the model with steps $(-1, -1, -1), (-1, -1, 1), (-1, 1, 0), (1, 0, 0)$. In [10, 4.3] it is shown that we have

$$Q(x, y, z; t) = \left[x^{>0}\right]\left[y^{>0}\right]\left[z^{>0}\right] \frac{\left(x - x^{-1}y - x^{-1}y^{-1}z - x^{-1}y^{-1}z^{-1}\right)\left(y - y^{-1}z - y^{-1}z^{-1}\right)\left(z - z^{-1}\right)}{xyz\left(1 - tS(x, y, z)\right)},$$

with the step counting polynomial

$$S(x, y, z) = x^{-1}y^{-1}z^{-1} + x^{-1}y^{-1}z + x^{-1}y + x.$$

We can find the dominant saddle point of $S(x, y, z)$ to be at $s_0 = \left(2^{3/4}, 2^{1/2}, 1\right)$, with 7 others associated to it. Using the notation as in (26), we can check that

$$\gamma = 2 \cdot 2^{3/4}, \quad c = \frac{7}{2},$$

---

[1]This is in fact a consequence of $K(x, y)$ still being quadratic in y, which allows one to argue in a fashion similar as in the proof of Thm. 1.

and that we have

$$v_1(k,l,m) = \frac{2^6}{\pi^{3/2}} 2^{-3k/4-l/2}(1+k)(1+l)(1+m),$$

$$v_2(k,l,m) = -\frac{2^4}{\pi^{3/2}} 2^{-3k/4-l/2}(1+k)(1+l)(1+m)(63 - 8k + 2k^2 - 4l + 4l^2 + 16m + 8m^2),$$

$$v_3(k,l,m) = \frac{2}{\pi^{3/2}} 2^{-3k/4-l/2}(1+k)(1+l)(1+m)(5313 - 32k^3 + 4k^4 - 32l^3 + 16l^4 + 3040m$$
$$+ 1776m^2 + 256m^3 + 64m^4 - 32k(43 - 3l + 3l^2 + 12m + 6m^2)$$
$$+ 8l^2(93 + 16m + 8m^2) + 4k^2(99 - 4l + 4l^2 + 16m + 8m^2) - 8l(103 + 48m + 24m^2)).$$

C   Decomposition of polyharmonic functions

In this section we will, as in Sec. 4.2, once again give the expressions after the substitution $k \mapsto k-1$, $l \mapsto l-1$ etc.

C.1   Gouyou-Beauchamps

For the Gouyou-Beauchamps model with steps $\mathcal{S} = \{\leftarrow, \rightarrow, \nwarrow, \searrow\}$, we obtain as a basis of the poly-harmonic functions (from here on, all functions will be given only up to constant multiples):

$$h_1^1 = kl(k+l)(k+2l),$$

$$h_1^2 = kl(k+l)(k+2l)(42 - 7k^2 + k^4 - 14kl + 4k^3l - 14l^2 - 8kl^3 - 4l^4),$$

$$h_1^3 = kl(k+l)(k+2l)(52976 - 11880k^2 + 2211k^4 - 110k^6 + 3k^8 - 23760kl + 8844k^3l - 660k^5l + 24k^7l$$
$$- 23760l^2 + 4620k^2l^2 - 1100k^4l^2 + 32k^6l^2 - 8448kl^3 - 144k^5l^3 - 4224l^4 + 2200k^2l^4 - 424k^4l^4$$
$$+ 2640kl^5 - 288k^3l^5 + 880l^6 + 128k^2l^6 + 192kl^7 + 48l^8),$$

$$h_2^1 = kl(k+l)(k+2l)(-20 + k^2 + 2kl + 2l^2),$$

$$h_2^2 = kl(k+l)(k+2l)(-3010 + 567k^2 - 78k^4 + k^6 + 1134kl - 312k^3l + 6k^5l + 1134l^2 - 84k^2l^2$$
$$+ 10k^4l^2 + 456kl^3 + 228l^4 - 20k^2l^4 - 24kl^5 - 8l^6),$$

$$h_3^1 = kl(k+l)(k+2l)(-373 + k^2 + 3k^4 + 2kl + 12k^3l + 2l^2 - 4k^2l^2 - 32kl^3 - 16l^4).$$

In this case, due to the symmetry of the model we can pick $\tilde{h}_n^m(u,v) := h_n^m(u,v)$. We consider once again the first three $v_p(k,l,u,v)$ as appearing in Thm. 2. We obtain

$$v_1(k,l,u,v) = kl(k+l)(k+2l)uv(u+v)(u+2v),$$

$$v_2(k,l,u,v) = kl(k+l)(k+2l)uv(u+v)(u+2v)(15 + 2k^2 + 4kl + 4l^2 + 2u^2 + 4uv + 4v^2),$$

$$v_3(k,l,u,v) = kl(k+l)(k+2l)uv(u+v)(u+2v)(1225 + 420k^2 + 20k^4 + 840kl + 80k^3l + 840l^2$$
$$+ 160k^2l^2 + 160kl^3 + 80l^4 + 420u^2 + 48k^2u^2 + 96klu^2 + 96l^2u^2 + 20u^4 + 840uv$$
$$+ 96k^2uv + 192kluv + 192l^2uv + 80u^3v + 840v^2 + 96k^2v^2 + 192klv^2 + 192l^2v^2$$
$$+ 160u^2v^2 + 160uv^3 + 80v^4).$$

We obtain

$$v_1 = h_1^1 \tilde{h}_1^1,$$

$$v_2 = 2\left(h_2^1 \tilde{h}_1^1 + h_1^1 \tilde{h}_2^1\right) + 95 h_1^1 \tilde{h}_1^1,$$

$$v_3 = -40\left(h_3^1 \tilde{h}_1^1 + h_1^1 \tilde{h}_3^1\right) + 48 h_2^1 \tilde{h}_2^1 + 2400\left(h_2^1 \tilde{h}_1^1 + h_1^1 \tilde{h}_2^1\right) + 140\left(h_1^2 \tilde{h}_1^1 + h_1^1 \tilde{h}_1^2\right) + 36425 h_1^1 \tilde{h}_1^1.$$

Here we notice that the number of summands is noticably reduced: as the degree of h_i^j increases by 4 whenever j increases by one, there are a lot less suitable combinations of base functions.

24

C.2 Tandem Walk

For the Tandem Walk, with steps $\mathcal{S} = \{\rightarrow, \downarrow, \nwarrow\}$, we have

$$
\begin{aligned}
v_1(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v), \\
v_2(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v)(15 + 2k^2 + 4kl + 4l^2 + 2u^2 + 4uv + 4v^2), \\
v_3(k,l,u,v) =& kl(k+l)(k+2l)uv(u+v)(u+2v)(1225 + 420k^2 + 20k^4 + 840kl + 80k^3l + 840l^2 \\
& + 160k^2l^2 + 160kl^3 + 80l^4 + 420u^2 + 48k^2u^2 + 96klu^2 + 96l^2u^2 + 20u^4 + 840uv \\
& + 96k^2uv + 192kluv + 192l^2uv + 80u^3v + 840v^2 + 96k^2v^2 + 192klv^2 + 192l^2v^2 \\
& + 160u^2v^2 + 160uv^3 + 80v^4).
\end{aligned}
$$

We can compute

$$
\begin{aligned}
v_1(k,l,u,v) =& kluv(k+l)(u+v), \\
v_2(k,l,u,v) =& kl(k+l)uv(u+v)(20 + 3k^2 + 3kl + 3l^2 + 3u^2 + 3uv + 3v^2), \\
v_3(k,l,u,v) =& kl(k+l)uv(u+v)(881 + 375k^2 + 12k^3 + 18k^4 + 375kl + 18k^2l + 36k^3l + 375l^2 - 18kl^2 \\
& + 54k^2l^2 - 12l^3 + 36kl^3 + 18l^4 + 375u^2 + 45k^2u^2 + 45klu^2 + 45l^2u^2 - 12u^3 + 18u^4 \\
& + 375uv + 45k^2uv + 45kluv + 45l^2uv - 18u^2v + 36u^3v + 375v^2 + 45k^2v^2 + 45klv^2 \\
& + 45l^2v^2 + 18uv^2 + 54u^2v^2 + 12v^3 + 36uv^3 + 18v^4).
\end{aligned}
$$

We can check that, although the model is not symmetric, we can let $\tilde{h}_n^m(u,v) := h_n^m(v,u)$, and obtain

$$
\begin{aligned}
v_1 =& h_1^1 \tilde{h}_1^1, \\
v_2 =& 3\left(h_2^1 \tilde{h}_1^1 + h_1^1 \tilde{h}_2^1\right) + 38 h_1^1 \tilde{h}_1^1, \\
v_3 =& 18\left(h_3^1 \tilde{h}_1^3 + h_1^1 \tilde{h}_3^1\right) + 45 h_2^1 \tilde{h}_2^1 + 660\left(h_2^1 \tilde{h}_1^1 + h_1^1 \tilde{h}_2^1\right) + \frac{10}{3}\left(h_1^2 \tilde{h}_1^1 + h_1^1 \tilde{h}_1^2\right) + 14780 h_1^1 \tilde{h}_1^1.
\end{aligned}
$$

Note that even though we have $v_3(k,l,1,1) \neq v_3(1,1,k,l)$ (i.e. we cannot just exchange starting and endpoint and hope to obtain the same number of paths), the resulting representation for v_3 in terms of products of the base functions is symmetrical. This is, as mentioned in the remark after Thm. 3, still due to a symmetry property, using the fact that the polyharmonic and adjoint polyharmonic functions are the same: the number of paths from a point x to a point y is forcibly the same as the number of paths from y to x with the reversed step set.

D Table of polyharmonic functions

This section contains the first three asymptotic terms of the 19 orbit-summable examples from [15]. Models 1 to 16 correspond to the models in [15, Table 1], models 17 and 18 to the first two in [15, Table 2], and finally model 19 is the same as the first one in [15, Table 3]. The remaining for models do not satisfy (17). All these models have, according to Thm. 1, asymptotics of the form

$$
q(k,l;n) = \frac{a}{\pi} \cdot \frac{\gamma^n}{n^c} \left[\sum_{p=1}^{3} \frac{v_p(k,l)\sum_{i=1}^{m} \alpha_i^{-k}\beta_i^{-l}\zeta_i^n}{n^p} + \mathcal{O}\left(\frac{1}{n^4}\right)\right]. \tag{73}
$$

Here, a is a multiplicative constant (which one could just as well move into the v_p, but is separate here to keep things shorter). The constants α, β, γ are defined as in Section 2, that is, given our dominant saddle point (x_0, y_0) with $x_0, y_0 > 0$, and another saddle point (x_i, y_i) we have $\alpha_i = x_i/x_0$, $\beta_i = y_i/y_0$ and $\zeta_i = S(x_i, y_i)/S(x_0, y_0)$. In particular, they all have modulus 1. The number of relevant saddle points is given by m. Also, it will be noted for each example if and when the cancellations occur that lead to the coefficients to vanish, i.e. for which constellations of n, k and l we will have $q(k,l;n) = 0$ (which is also a direct consequence of (73) and can be checked to agree with basic combinatorial considerations).

D.1 Model 1 (Simple Walk)

$\mathcal{S} = x + y + x^{-1} + y^{-1}$.
Relevant saddle points: $(1,1), (-1,-1)$.
$\gamma = 4$,
$c = 2$,
$a = 16$.
Periodicity: $q(k,l,n)$ vanishes if $j + k \not\equiv n \mod 2$.

$$v_1(k,l) = (1+k)(1+l),$$

$$v_2(k,l) = -\frac{1}{2}(1+k)(1+l)(15 + 2k(2+k) + 2l(2+l)),$$

$$v_3(k,l) = \frac{1}{4}(1+k)(1+l)(317 + 168l + 4(k(2+k)(21 + k(2+k)) + 4k(2+k)l$$
$$+ (25 + 2k(2+k))l^2 + 4l^3 + l^4)).$$

D.2 Model 2 (Diagonal Walk)

$\mathcal{S} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1}$.
Relevant saddle points: $(1,1), (-1,-1), (1,-1), (-1,1)$.
$\gamma = 4$,
$c = 2$,
$a = 2$.
Periodicity: $q(k,l,n)$ vanishes if $j + k \not\equiv n + k$ or $j + k \not\equiv \mod 2$.

$$v_1(k,l) = (1+k)(1+l),$$

$$v_2(k,l) = -\frac{1}{2}(1+k)(1+l)(9 + k(2+k) + l(2+l)),$$

$$v_3(k,l) = \frac{1}{4}(1+k)(1+l)(113 + 4k^3 + k^4 + 4k(13 + l(2+l))$$
$$+ 2k^2(15 + l(2+l)) + l(2+l)(26 + l(2+l))).$$

D.3 Model 3

$\mathcal{S} = xy + y + x^{-1}y + x^{-1}y + x^{-1} + x^{-1}y^{-1}$.
Relevant saddle points: $(1,1), (1,-1)$.
$\gamma = 6$,
$c = 2$,
$a = \frac{3\sqrt{6}}{2}$.
Periodicity: $q(k,l;n)$ vanishes if $l \not\equiv n \mod 2$.

$$v_1(k,l) = \frac{2\sqrt{6}}{\pi}(1+k)(1+l),$$

$$v_2(k,l) = -\frac{1}{16}(1+k)(1+l)(75 + 12k(2+k) + 8l(2+l)),$$

$$v_3(k,l) = \frac{1}{512}(1+k)(1+l)(7793 + 3424l + 8(3k(2+k)(103 + 6k(2+k)) + 48k(2+k)l$$
$$+ 6(41 + 4k(2+k))l^2 + 32l^3 + 8l^4)).$$

D.4 Model 4 (King's Walk)

$\mathcal{S} = xy + y + x^{-1}y + x + x^{-1} + xy^{-1} + x^{-1} + x^{-1}y^{-1}$.
Relevant saddle point: $(1,1)$.
$\gamma = 8$,

$c = 2,$
$a = \frac{128}{27}.$

$$v_1(k,l) = (1+k)(1+l),$$
$$v_2(k,l) = -\frac{1}{3}(1+k)(1+l)(15 + 2k(2+k) + 2l(2+l)),$$
$$v_3(k,l) = \frac{1}{18}(1+k)(1+l)(307 + 168l + 4(k(2+k)(21 + k(2+k))$$
$$+ 4k(2+k)l + (25 + 2k(2+k))l^2 + 4l^3 + l^4)).$$

D.5 Model 5

$\mathcal{S} = xy + x^{-1}y + y^{-1}.$
Relevant saddle points: $\left(1, \frac{1}{\sqrt{2}}\right), \left(1, -\frac{1}{\sqrt{2}}\right), \left(-1, \frac{i}{\sqrt{2}}\right), \left(-1, \frac{-i}{\sqrt{2}}\right).$
$\gamma = 2\sqrt{2},$
$c = 2,$
$a = 4\sqrt{2}.$
Periodicity: $q(k,l;n)$ vanishes if $n + l \equiv 0 \mod 4 \wedge k \equiv 0 \mod 2$ or $n + l \equiv 2 \mod 4 \wedge k \equiv 1 \mod 2.$

$$v_1(k,l) = 2^{l/2}(1+k)(1+l),$$
$$v_2(k,l) = -\frac{1}{4}2^{l/2}(1+k)(1+l)(27 + 8k + 4k^2 + 10l + 2l^2),$$
$$v_3(k,l) = \frac{1}{32}2^{l/2}(1+k)(1+l)(1049 + 64k^3 + 16k^4 + 748l + 264l^2 + 40l^3$$
$$+ 4l^4 + 16k(43 + 14l + 2l^2) + 8k^2(51 + 14l + 2l^2)).$$

D.6 Model 6

$\mathcal{S} = xy + x^{-1}y + y^{-1} + x + x^{-1}.$
Relevant saddle point: $\left(1, \frac{1}{\sqrt{2}}\right).$
$\gamma = 2 + 2\sqrt{2},$
$c = 2,$
$a = 2\sqrt{7 + 5\sqrt{2}}.$

$$v_1(k,l) = 2^{l/2}(1+k)(1+l),$$
$$v_2(k,l) = -\frac{-239 + 169\sqrt{2}}{2}2^{l/2}(1+k)(1+l)(116592 + 82443\sqrt{2}$$
$$+ 4(3363 + 2378\sqrt{2})k(2+k) + 2l(26327 + 18616\sqrt{2} + (8119 + 5741\sqrt{2})l)),$$
$$v_3(k,l) = \frac{-47321 + 33461\sqrt{2}}{16}2^{l/2}(787537656 + 556873217\sqrt{2}$$
$$+ 32(941664 + 665857\sqrt{2})k^3 + 8(941664 + 665857\sqrt{2})k^4$$
$$+ 16k(24218939 + 17125376\sqrt{2} + l(9255076 + 6544327\sqrt{2}(2273378 + 1607521\sqrt{2})l))$$
$$+ 8k^2(27985595 + 19788804\sqrt{2} + l(9255076 + 6544327\sqrt{2} + (2273378 + 1607521\sqrt{2})l))$$
$$+ 2l(345292494 + 244158664\sqrt{2} + l(168998590 + 119500049\sqrt{2}$$
$$+ l(35593948 + 25168722\sqrt{2} + (5488420 + 3880899\sqrt{2})l)))).$$

D.7 Model 7

$\mathcal{S} = y + y^{-1} + xy + x^{-1}y.$
Relevant saddle points: $\left(1, \frac{1}{\sqrt{3}}\right), \left(1, -\frac{1}{\sqrt{3}}\right).$

$\gamma = 2\sqrt{3},$
$c = 2,$
$a = 6\sqrt{3}.$
Periodicity: $q(k, l; n)$ vanishes if if $n + l \not\equiv 0 \mod 2$.

$$v_1(k, l) = 3^{l/2}(1 + k)(1 + l),$$

$$v_2(k, l) = -\frac{1}{8}3^{l/2}(1 + k)(1 + l)(57 + 24k + 12k^2 + 20l + 4l^2),$$

$$v_3(k, l) = \frac{1}{128}3^{l/2}(1 + k)(1 + l)(4649 + 576k^3 + 144k^4 + 3160l + 1080l^2 + 160l^3$$
$$+ 16l^4 + 48k(85 + 28l + 4l^2) + 24k^2(109 + 28l + 4l^2)).$$

D.8 Model 8

$S = xy + y + x^{-1}y + x + x^{-1} + y^{-1}.$
Relevant saddle point: $\left(1, \frac{1}{\sqrt{3}}\right).$
$\gamma = 2 + 2\sqrt{3},$
$c = 2,$
$a = 2\sqrt{10 + 6\sqrt{3}}.$

$$v_1(k, l) = 3^{l/2}(1 + k)(1 + l),$$

$$v_2(k, l) = \frac{-332313 + 191861\sqrt{3}}{48}3^{l/2}(1 + k)(1 + l)(16850187 + 9728460\sqrt{3}$$
$$+ 12(191861 + 110771\sqrt{3})k(2 + k) + 4l(1623931 + 937577\sqrt{3} + (524174 + 302632\sqrt{3})l)),$$

$$v_3(k, l) = \frac{897909603 - 518408351\sqrt{3}}{4608}3^{l/2}(1 + k)(1 + l)$$
$$\cdot (3(6408751361721 + 3700094323859\sqrt{3}) + 1152(897909603 + 518408351\sqrt{3})k^3$$
$$+ 288(897909603 + 518408351\sqrt{3})k^4 + 16(925894850856 + 534565641383\sqrt{3})l$$
$$+ 16(434596264689 + 250914270407\sqrt{3})l^2 + 128(10381790502 + 5993929541\sqrt{3})l^3$$
$$+ 64(3351044259 + 1934726305\sqrt{3})l^4 + 96k(113649411684 + 65615518429\sqrt{3}$$
$$+ 4(9395817327 + 5424677663\sqrt{3})l + 8(1226567328 + 708158977\sqrt{3})l^2)$$
$$+ 48k^2(135199242156 + 78057318853\sqrt{3} + 4(9395817327 + 5424677663\sqrt{3})l$$
$$+ 8(1226567328 + 708158977\sqrt{3})l^2)).$$

D.9 Model 9

$S = xy + y + x^{-1}y + x^{-1}y^{-1} + xy^{-1}.$
Relevant saddle points: $\left(1, \sqrt{\frac{2}{3}}\right), \left(1, -\sqrt{\frac{2}{3}}\right).$
$\gamma = 2\sqrt{6},$
$c = 2,$
$a = \frac{12\sqrt{6}}{5\sqrt{5}}.$

Periodicity: $q(k,l;n)$ is non-zero if $l + n \equiv 0 \mod 2$.

$$v_1(k,l) = \left(\frac{3}{2}\right)^{l/2} (1+k)(1+l),$$

$$v_2(k,l) = -\frac{1}{2}\left(\frac{3}{2}\right)^{l/2}(1+k)(1+l)(93 + 24k + 12k^2 + 14l + 10l^2),$$

$$v_3(k,l) = \frac{1}{20000}\left(\frac{3}{2}\right)^{l/2}(1+k)(1+l)(59581 + 2880k^3 + 720k^4 + 20540l + 14400l^2 + 1400l^3$$
$$+ 500l^4 + 240k(133 + 10l + 10l^2) + 120k^2(157 + 10l + 10l^2)).$$

D.10 Model 10

$\mathcal{S} = xy + y + x^{-1}y + x + x^{-1} + xy^{-1} + x^{-1}y^{-1}$.
Relevant saddle point: $\left(1, \sqrt{\frac{2}{3}}\right)$.
$\gamma = 2 + 2\sqrt{6}$,
$c = 2$,
$a = \frac{2\sqrt{(15893 + 4923\sqrt{6})}}{19\sqrt{19}}$.

$$v_1(k,l) = \left(\frac{3}{2}\right)^{l/2}(1+k)(1+l),$$

$$v_2(k,l) = -\frac{\sqrt{1/5(-19+9\sqrt{6})}}{(21660(1+\sqrt{6})^{3/2}(4027973401873 + 1644413252328\sqrt{6})}\left(\frac{3}{2}\right)^{l/2}(1+k)(1+l)$$
$$\cdot (45(18505016772606410202221 + 7554641462421834757881\sqrt{6})$$
$$+ 114(84987940228353237925 + 34696181308271177306\sqrt{6})k(2+k)$$
$$+ 19l(10112697572977267931562 + 4128491496146047169957\sqrt{6}$$
$$+ 19(33321936262704908072 + 13603623514195217076\sqrt{6})l)),$$

$$v_3(k,l) = \frac{\sqrt{-95+45\sqrt{6}}}{312770400(1+\sqrt{6})^{39/2}(7+2\sqrt{6})^5(2390878567 + 976071762\sqrt{6})}\left(\frac{3}{2}\right)^{l/2}(1+k)(1+l)$$
$$\cdot (2278547431987218871699749243102621 + 93021309384960729142773509537545\sqrt{6}$$
$$+ 190(587149712722343534306336232584 + 23970286646524502952271000638793\sqrt{6})l$$
$$+ 38(2736(8664069065571669392670303409 + 35370913844804731165412950\sqrt{6})k^3$$
$$+ 684(8664069065571669392670303409 + 35370913844804731165412950\sqrt{6})k^4$$
$$+ 19l^2(7(15000641943020052698954122627 + 6123986429098456970720167622\sqrt{6})$$
$$+ 19l(80841533399881759930812247918 + 33003417808979012226614318448\sqrt{6}$$
$$+ 19(13318881554059560613996041 + 5437410625335494751786237\sqrt{6})l))$$
$$+ 12k(242183848551316225728154335258 + 9887114214903395698238202393071\sqrt{6}$$
$$+ 38l(8576544194162402436184705763 + 35013595053545704368047082063\sqrt{6}$$
$$+ 19(3396994401828483455926278281 + 13868171572617913144935146\sqrt{6})l))$$
$$+ 6k^2(5(5633840069806460763174618376078 + 23000055772452982878762065870\sqrt{6})$$
$$+ 38l(8576544194162402436184705763 + 35013595053545704368047082063\sqrt{6}$$
$$+ 19(3396994401828483455926278281 + 13868171572617913144935146\sqrt{6})l))))).$$

D.11 Model 11

$\mathcal{S} = y + xy^{-1} + y^{-1} + x^{-1}y^{-1}.$
Relevant saddle points: $\left(1, \sqrt{3}\right), \left(1, -\sqrt{3}\right).$
$\gamma = 2\sqrt{3},$
$c = 2,$
$a = 6\sqrt{3}.$
Periodicity: $q(k, l; n)$ is non-zero if $l + n \equiv 0 \mod 2.$

$$
\begin{aligned}
v_1(k, l) =& 3^{-l/2}(1 + k)(1 + l), \\
v_2(k, l) =& -\frac{1}{8}3^{-l/2}(1 + k)(1 + l)(57 + 12k(2 + k) + 4(-1 + l)l), \\
v_3(k, l) =& \frac{1}{128}3^{-l/2}(1 + k)(1 + l)(4649 - 824l + 8(3k(2 + k)(85 + 6k(2 + k)) - 36k(2 + k)l \\
& + 3(29 + 4k(2 + k))l^2 - 4l^3 + 2l^4)).
\end{aligned}
$$

D.12 Model 12

$\mathcal{S} = y + xy^{-1} + y^{-1} + x^{-1}y^{-1} + x + x^{-1}.$
Relevant saddle point: $\left(1, \sqrt{3}\right).$
$\gamma = 2 + 2\sqrt{3},$
$c = 2,$
$a = 2\sqrt{10 + 6\sqrt{3}}.$

$$
\begin{aligned}
v_1(k, l) =& 3^{-l/2}(1 + k)(1 + l), \\
v_2(k, l) =& \frac{191861\sqrt{3} - 332313}{48}3^{-l/2}(1 + k)(1 + l)(16850187 + 9728460\sqrt{3} + 12(191861 \\
& + 110771\sqrt{3})k(2 + k) + 4l(472765 + 272951\sqrt{3} + (524174 + 302632\sqrt{3})l)), \\
v_3(k, l) =& \frac{897909603 - 518408351\sqrt{3}}{4608}3^{-l/2}(1 + k)(1 + l)(3(6408751361721 + 3700094323859\sqrt{3}) \\
& + 1152(897909603 + 518408351\sqrt{3})k^3 + 288(897909603 + 518408351\sqrt{3})k^4 \\
& + 96k(113649411684 + 65615518429\sqrt{3} + 4l(416721297 + 240594153\sqrt{3} \\
& + 2(1226567328 + 708158977\sqrt{3})l)) + 48k^2(135199242156 + 78057318853\sqrt{3} \\
& + 4l(416721297 + 240594153\sqrt{3} + 2(1226567328 + 708158977\sqrt{3})l)) \\
& + 16l(254773295988 + 147093431021\sqrt{3} + l(316845801201 + 182931008615\sqrt{3} \\
& + 4l(6044773068 + 3489951358\sqrt{3} + (3351044259 + 1934726305\sqrt{3})l)))).
\end{aligned}
$$

D.13 Model 13

$\mathcal{S} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1} + y^{-1}.$
Relevant saddle points: $\left(1, \sqrt{\frac{3}{2}}\right), \left(1, -\sqrt{\frac{3}{2}}\right).$
$\gamma = 2\sqrt{6},$
$c = 2,$
$a = \frac{12\sqrt{6}}{5\sqrt{5}}.$

Periodicity: $q(k, l; n)$ is non-zero if $n + l \equiv 0 \mod 2$.

$$v_1(k, l) = \left(\frac{2}{3}\right)^{l/2} (1+k)(1+l),$$

$$v_2(k, l) = -\frac{1}{20} \left(\frac{2}{3}\right)^{l/2} (1+k)(1+l)(93 + 12k(2+k) + 2l(13 + 5l)),$$

$$v_3(k, l) = \frac{1}{4000} \left(\frac{2}{3}\right)^{l/2} (1+k)(1+l)(59581 + 32660l + 20(6k(2+k)(133 + 6k(2+k))$$
$$+ 180k(2+k)l + 60(14 + k(2+k))l^2 + 130l^3 + 25l^4)).$$

D.14   Model 14

$\mathcal{S} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1} + x + x^{-1} + y^{-1}$.
Relevant saddle point: $\left(1, \sqrt{\frac{3}{2}}\right)$.
$\gamma = 2 + 2\sqrt{6}$,
$c = 2$,
$a = \frac{15893 + 4923\sqrt{6}}{19\sqrt{19}}$.

$$v_1(k, l) = \left(\frac{2}{3}\right)^{l/2} (1+k)(1+l),$$

$$v_2(k, l) = -\frac{\sqrt{5}}{4332(1+\sqrt{6})^{23/2}(119287 + 48682\sqrt{6})} \left(\frac{2}{3}\right)^{l/2} (1+k)(1+l)$$
$$\cdot (45(923103972550017581 + 376855618721953841\sqrt{6})$$
$$+ 114(42395371053890538 + 17307837756843143\sqrt{6})k(2+k)$$
$$+ 19l(758833876782694710 + 309792632946741935\sqrt{6} + 19(16622309568974292$$
$$+ 6786029465163487\sqrt{6})l)),$$

$$v_3(k, l) = \frac{\sqrt{5}(847 + 342\sqrt{6})}{12510816(1+\sqrt{6})^{39}2(7+2\sqrt{6})^5(2390878567 + 976071762\sqrt{6})} \left(\frac{2}{3}\right)^{l/2} (1+k)(1+l)$$
$$(22785474319872188716997492431026219 + 93021309384960729148277350953754 59\sqrt{6}$$
$$+ 38(40727864154075605264559945061147 4 + 166270809151458123021641573213639\sqrt{6})l$$
$$+ 38(2736(8664069065571669392670303409 + 353709138448047311654129504 9\sqrt{6})k^3$$
$$+ 684(8664069065571669392670303409 + 353709138448047311654129504 9\sqrt{6})k^4$$
$$+ 19l^2(120494788073478253451702687350 43 + 4919179124080296718913160023698\sqrt{6}$$
$$+ 19l(12160546622182356140192758729 0 + 4964522369612050800053649444 0\sqrt{6}$$
$$+ 19(1331888155405956061399604179 + 543741062533549475178623769\sqrt{6})l))$$
$$+ 12k(242183848551316225728154335258 86 + 988711421490339569823820239307 1\sqrt{6}$$
$$+ 38l(17240613259734071828855009172 3 + 70384508898350435533460032553\sqrt{6}$$
$$+ 19(3396994401828483455926278281 + 1386817157261791314493514666\sqrt{6})l))$$
$$+ 6k^2(5(56338400698064607631746183760 78 + 2300005577245298287876206587083\sqrt{6})$$
$$+ 38l(17240613259734071828855009172 3 + 70384508898350435533460032553\sqrt{6}$$
$$+ 19(3396994401828483455926278281 + 1386817157261791314493514666\sqrt{6})l)))).$$

D.15 Model 15

$\mathcal{S} = y + xy^{-1} + x^{-1}y^{-1}$.
Relevant saddle points: $\left(1, \sqrt{2}\right), \left(1, -\sqrt{2}\right), \left(-1, i\sqrt{2}\right), \left(-1, -i\sqrt{2}\right)$.
$\gamma = 2\sqrt{2}$,
$c = 2$,
$a = 4\sqrt{2}$.
Periodicity: $q(k,l;n)$ is non-zero if $n + l \equiv 0 \mod 4 \wedge k \equiv 1 \mod 2$ or $n + l \equiv 2 \mod 3 \wedge k \equiv 0 \mod 2$.

$$v_1(k,l) = 2^{-l/2}(1+k)(1+l),$$
$$v_2(k,l) = -\frac{1}{4}2^{-l/2}(1+k)(1+l)27 + 4k(2+k) + 2(-1+l)l),$$
$$v_3(k,l) = \frac{1}{32}2^{-l/2}(1+k)(1+l)(1049 + 8k(2+k)(43 + 2k(2+k)) - 188l - 48k(2+k)l$$
$$+ 8(21 + 2k(2+k))l^2 - 8l^3 + 4l^4).$$

D.16 Model 16

$\mathcal{S} = y + x + x^{-1} + xy^{-1} + x^{-1}y^{-1}$.
Relevant saddle point: $\left(1, \sqrt{2}\right)$.
$\gamma = 2 + 2\sqrt{2}$,
$c = 2$,
$a = 2\sqrt{7 + 5\sqrt{2}}$.

$$v_1(k,l) = 2^{-l/2}(1+k)(1+l),$$
$$v_2(k,l) = \frac{4756 - 3363\sqrt{2}}{8}2^{-l/2}(1+k)(1+l)(116592 + 82443\sqrt{2}$$
$$+ 4(3363 + 2378\sqrt{2})k(2+k) + 2l(6149 + 4348\sqrt{2} + (8119 + 5741\sqrt{2})l)),$$
$$v_3(k,l) = \frac{-941664 + 665857\sqrt{2}}{64}2^{-l/2}(1+k)(1+l)(787537656 + 556873217\sqrt{2}$$
$$+ 32(941664 + 665857\sqrt{2})k^3 + 8(941664 + 665857\sqrt{2})k^4 + 16k(24218939$$
$$+ 17125376\sqrt{2} + l(-161564 - 114243\sqrt{2} + (2273378 + 1607521\sqrt{2})l))$$
$$+ 8k^2(27985595 + 19788804\sqrt{2} + l(-161564 - 114243\sqrt{2} + (2273378 + 1607521\sqrt{2})l))$$
$$+ 2l(77265162 + 54634720\sqrt{2} + l(114437518 + 80919545\sqrt{2} + l(8313412 + 5878470\sqrt{2}$$
$$+ (5488420 + 3880899\sqrt{2})l)))).$$

D.17 Model 17 (Tandem Walk)

$\mathcal{S} = y + x^{-1} + xy^{-1}$.
Relevant saddle points: $(1,1), \left(e^{4\pi i/3}, e^{2\pi i/3}\right), \left(e^{2\pi i/3}, e^{4\pi i/3}\right)$.
$\gamma = 3$,
$c = 3$,
$a = \frac{27\sqrt{3}}{2}$.
Periodicity: $q(k,l;n)$ is non-zero if $2k + l \equiv n \mod 3$.

$$v_1(k,l) = (1+k)(1+l)(2+k+l),$$
$$v_2(k,l) = -\frac{1}{3}(1+k)(1+l)(2+k+l)(38 + 3k^2 + 9l + 3l^2 + 3k(3+l)),$$
$$v_3(k,l) = \frac{1}{18}(1+k)(1+l)(2+k+l)(1930 + 9k^4 + 954l + 417l^2 + 60l^3 + 9l^4$$
$$+ 6k^3(8 + 3l) + 3k^2(121 + 33l + 9l^2) + 3k(300 + 157l + 39l^2 + 6l^3)).$$

D.18 Model 18

$\mathcal{S} = y + x + xy^{-1} + y^{-1} + x^{-1} + x^{-1}y.$
Relevant saddle point: $(1, 1)$.
$\gamma = 6,$
$c = 3,$
$a = \frac{27\sqrt{3}}{2}.$

$$v_1(k, l) = (1 + k)(1 + l)(2 + k + l),$$
$$v_2(k, l) = -(1 + k)(1 + l)(2 + k + l)(11 + k^2 + k(3 + l) + l(3 + l)),$$
$$v_3(k, l) = \frac{9}{104}(1 + k)(1 + l)(2 + k + l)(940 + 6k^4 + 12k^3(3 + l) + 3l(3 + l)(57 + 2l(3 + l))$$
$$+ 3k(3 + l)(57 + 4l(3 + l)) + 9k^2(25 + 2l(4 + l))).$$

D.19 Model 19 (Gouyou-Beauchamps)

$\mathcal{S} = x + x^{-1} + xy^{-1} + x^{-1}y.$
Relevant saddle points: $(1, 1), (-1, 1)$.
$\gamma = 4,$
$c = 4,$
$a = 64.$
Periodicity: $q(k, l; n)$ is non-zero if $k + n \equiv 0 \mod 2$.

$$v_1(k, l) = (1 + k)(1 + l)(2 + k + l)(3 + k + 2l),$$
$$v_2(k, l) = -\frac{1}{2}(1 + k)(1 + l)(2 + k + l)(3 + k + 2l)(35 + 2k^2 + 4k(2 + l) + 4l(3 + l)),$$
$$v_3(k, l) = \frac{1}{8}(1 + k)(1 + l)(2 + k + l)(3 + k + 2l)(25 + 2k^2 + 4k(2 + l)$$
$$+ 4l(3 + l))(61 + 2k^2 + 4k(2 + l) + 4l(3 + l)).$$